# IDEMPOTENT ELEMENTS IN BLOCKS OF $p$-SOLVABLE GROUPS

By

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To my grandmother

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Given a finite group $G$ and an algebraically closed field $F$ of prime characteristic $p$, the $p$-blocks of $G$ are the indecomposable two-sided ideals of the group algebra $FG$. Each block of $G$ is determined by a primitive idempotent of the center $Z(FG)$ of the group algebra.

When studying a ring $R$ in general, one is interested in the category of $R$-modules. Given a finite dimensional algebra $A$, its basic algebra is another algebra $B$ which is simplest among those whose category of $B$-modules is equivalent to the category of $A$-modules.

In this work, we investigate properties of idempotent elements in blocks of $p$-solvable groups, and we describe a method for calculating the basic algebra of such a block. We conclude with a software implementation of this method, and we present some empirical data obtained using this implementation.

# CHAPTER 1
# INTRODUCTION

The study of discrete mathematical structures plays a crucial role in modern mathematics, not only because of its intrinsic interest, but also because of its applications to science. At the heart of the study of such structures is finite group theory. Groups are a mathematical realization of the notion of symmetry, and their study is a fundamental part of abstract algebra and mathematics in general. One way to study finite groups is to examine the manner in which they can act upon other various mathematical objects. Group representation theory is the study of actions of groups on vector spaces.

Modular representation theory is the study of representations of finite groups on finite dimensional vector spaces over fields of prime characteristic. Block theory is a tool for studying modular representations. In addition, block theory sheds light on the relationships between the modular representations of a given finite group and those in characteristic zero and often yields new information about both.

One way to understand the representation theory of any given finite dimensional associative algebra (or, for example, a group algebra of a finite group) is to understand its category of modules. Such categories were studied in [16]. In this paper, Morita proved that the isomorphism class of the basic algebra of an algebra $A$ determines the equivalence class of the category whose objects are $A$-modules and whose morphisms are $A$-module homomorphisms. This fact has inspired a great deal of current interest in the calculation of basic algebras.

In this dissertation, we provide a concrete treatment of block idempotents of $p$-solvable groups and a treatment of the basic algebra of a block of a $p$-solvable group at the level of idempotents and group characters. A main result is a method for computing the basic algebra of a block of a $p$-solvable group by systematically combining the theory of Morita in [16] and Fong's theory of characters and blocks of $p$-solvable groups, originally developed in [7]. We conclude by discussing an algorithm for computing the basic algebra of a block of $p$-solvable group and its implementation in the GAP [8] computer software.

In Chapter 2, we discuss the necessary preliminaries from the representation theory of finite groups. Section 2.1 is a recollection of standard facts about finite dimensional algebras, finite dimensional modules over algebras, and idempotent elements in algebras. We define the basic algebra of a finite dimensional algebra as in [1], and we show that the basic algebra of an algebra $A$ can be constructed using some idempotent element in $A$. Given algebras $A$ and $B$, we say that $A$ and $B$ are *Morita equivalent* if they have isomorphic basic algebras (and hence, $A$ and $B$ have equivalent module categories). Hence, Morita equivalence is a slightly weaker notion than that of isomorphism, but many interesting properties of algebras (e.g. simplicity, semisimplicity, etc.) are preserved by Morita equivalence. We also prove some standard theorems about the basic algebra at the level of idempotents. In section 2.2, we recall some facts and notation involving characters and blocks of finite groups. Our main reference for ordinary character theory and complex representation theory is [12], and our main reference for Brauer character theory, and modular representation theory, and block theory is [17]. In particular, we recall theorems due to Clifford, Green, and Fong-Reynolds involving the relationships between characters and blocks of a group and those of a normal subgroup. In [7], Paul Fong first studied the characters and blocks of $p$-solvable groups. Section 2.3 recalls the main results of the theory of characters and blocks of $p$-solvable groups, using [17, Chapter 10] as our main reference.

Chapter 3 is a study of block idempotents in $p$-solvable groups for blocks of full defect. In [19], Tiedt proved that if $B$ is a block of full defect (i.e. the defect groups of $B$ are precisely the Sylow $p$-subgroups of $G$), that the support of the block idempotent $e_B$ is contained in $O_{p'}(G)$, the largest normal subgroup of $G$ whose order is not divisible by $p$. In Section 3.1, we use this result to provide a calculation of the block idempotent $e_B$ of a block $B$ of full defect in terms of the Brauer characters which belong to $B$ and also in terms of block idempotents of blocks of $O_{p'}(G)$ which are covered by $B$. In Section 3.2, we specialize to blocks $B$ of $p$-solvable groups which contain a linear Brauer character. We prove that for such a block $B$, there exists a unique linear $G$-invariant character of $O_{p'}(G)$ that controls the structure of $B$. In particular, the block idempotent of $B$ is equal to the block idempotent corresponding to this character. We also give (in terms of the afforementioned character) an explicit isomorphism between $B$ and the principal block (the block which contains the trivial character). We conclude Section 3.2 with a theorem which counts the number of irreducible Brauer characters in a block whose only irreducible Brauer characters are linear.

Chapter 4 contains our theoretical study of the basic algebra block $B$ of a $p$-solvable group. Section 4.1 contains a theorem that yields a method for constructing an idempotent which determines the basic algebra of $B$ up to isomorphism. A slight drawback of this general result is that if $B$ contains an irreducible Brauer character whose degree has a nontrivial $p'$-part, then one must be able to find primitive (not necessarily central) idempotents in order to calculate the basic algebra explicitly. However, in the event that every Brauer character in $B$ has degree a power of $p$, the basic algebra can be constructed explicitly without searching for primitive idempotents. Section 4.2 is dedicated to proving a special case of a result mentioned in [2] which says that if $B$ is a nilpotent block of an arbitrary finite group $G$ and $B$ has abelian defect groups, then the basic algebra of $B$ is isomorphic to the group algebra over a defect group. We present a new proof of this result (using only elementary techniques of character theory and linear algebra) in the special case that $G$ is a $p$-nilpotent (and hence also $p$-solvable) group. Finally, in Section 4.3, we extend a result of Ninomaya and Wada in [18] to refine our calculation of the basic algebra in Section 4.1 in the case that $B$ contains exactly two Brauer characters, and at least one of these characters is linear.

Chapter 5 concerns the implementation in GAP of the method of computing the basic algebra of a block of a $p$-solvable group described in Section 4.1. GAP (which stands for Groups, Algorithms, and Programming) is a free and open-source computational discrete algebra system originally developed at Lehrstuhl D für Mathematik in 1986. Since then, GAP has gone through many different versions, and its development is currently jointly-coordinated by many different authors. Section 5.1 is a description of the algorithm, Section 5.2 is the GAP source code, and Section 5.3 contains some information about examples computed using this code. At the 2013 Joint Mathematics Meetings in San Diego, California, Klaus Lux presented in his talk [14] a GAP package which is used to study basic algebras of blocks of simple groups. Lux's algorithm uses results of a different nature such as those which appear in [15] and [3].

CHAPTER 2
PRELIMINARY REPRESENTATION THEORY

## 2.1 Algebras, Modules, and Idempotents

In this section, we recall some fundamental definitions and results about algebras, modules, and idempotents. Throughout, suppose that $F$ is an algebraically closed field and that $A$ is a finite dimensional, associative $F$-algebra with 1.

Since $A$ is finite dimensional, we can decompose the regular left $A$-module $A$ as a direct sum

$$_AA = \bigoplus_{i=1}^{n} A_i,$$

where the $A_i$ are indecomposable $A$-submodules, called *principal indecomposable modules* of $A$. Note that $A_i$ is an indecomposable left ideal of $A$. By the Krull-Schmidt theorem, the $A_i$ are unique up to isomorphism and order of occurrence. Thus, the principal indecomposable modules (sometimes called projective indecomposable modules or PIMs) of $A$ are defined up to isomorphism.

An element $e \in A$ is called an *idempotent* if $e^2 = e$. Two idempotents, $e, f \in A$ are *orthogonal* if $ef = 0 = fe$. An idempotent $e \in A$ is *primitive* if $e \neq 0$ and $e$ can not be written as the sum of two nonzero orthogonal idempotents.

There is an inherent connection between primitive idempotents and principal indecomposable modules. We will show, in Lemmas 2.1 and 2.2 below, that finding a set of primitive orthogonal idempotents whose sum is 1 in $A$ is equivalent to finding a decomposition of $A$ into a direct sum of principal indecomposable modules.

**Lemma 2.1.** *Suppose $A = \bigoplus_{i=1}^{n} A_i$ is a decomposition of $A$ into a direct sum of left ideals of $A$. For each $1 \leq i \leq n$, let $e_i$ be the unique element of $A_i$ such that $1 = e_1 + \cdots + e_n$. Then the $e_i$ are orthogonal idempotents of $A$ and $A_i = Ae_i$ for all $i$. Furthermore, if $A_i$ is indecomposable, then $e_i$ is a primitive idempotent.*

*Proof.* Since $1 = e_1 + \cdots + e_n$, we have that for each $j$, $e_j = \sum_{i=1}^{n} e_j e_i \in A_j$. Since $A = \bigoplus_{i=1}^{n} A_i$ is a direct sum, this implies that $e_j e_i = \delta_{ij} e_j$ for all $i, j$. Hence, the $e_i$ are mutually orthogonal idempotents. Now, if $a \in A_i$, then

$$a = a1 = a \sum_{j=1}^{n} e_j \in A_i,$$

and so $A_i e_j = 0$ if $i \neq j$, since the sum is direct. Now,

$$A_i = A_i 1 = A_i \left( \sum_{j=1}^{n} e_j \right) = A_i e_i$$

since $A_i e_j = 0$ if $i \neq j$.

Now, suppose that $A_i$ is indecomposable and that $e_i = f_i + g_i$ for nonzero orthogonal idempotents $f_i$ and $g_i$. Then

$$A_i = Ae_i = A(f_i + g_i) \subseteq Af_i + Ag_i.$$

Now, since $f_i$ and $g_i$ are orthogonal, if $x, y \in A$, then

$$xf_i + yg_i = xf_i(f_i + g_i) + yg_i(f_i + g_i) = (xf_i + yg_i)(f_i + g_i) \in A(f_i + g_i).$$

Hence, $A_i = Af_i + Ag_i$. Furthermore, if $x \in Af_i \cap Ag_i$, then $x = af_i$ for some $a \in A$. But since $x \in Ag_i$, we have that

$$x = xg_i = af_i g_i = 0.$$

Hence, $A_i = Af_i \oplus Ag_i$ is a direct sum of two nonzero left ideals of $A$, which is a contradiction. $\square$

**Lemma 2.2.** *Suppose that $e_1, \ldots, e_n \in A$ are orthogonal idempotents such that $1 = e_1 + \cdots + e_n$. Then $A = \bigoplus_{i=1}^{n} Ae_i$. Furthermore, if $e_i$ is a primitive idempotent for some $i$, then $Ae_i$ is an indecomposable left ideal.*

*Proof.* First, we have that

$$A = A1 = A(e_1 + \cdots + e_n) = Ae_1 + \cdots Ae_n.$$

If $a \in Ae_i$, then $a = ae_i$. If $a \in \sum_{\substack{j=1 \\ j \neq i}}^{n} Ae_j$, then

$$a = a(e_1 + \cdots + \widehat{e_i} + \cdots + e_n) = a(1 - e_i).$$

Thus, if $a \in Ae_i \cap \sum_{\substack{j=1 \\ j \neq i}}^{n} Ae_j$, then

$$a = ae_i = a(1 - e_i)e_i = a0 = 0.$$

Therefore, $A = \bigoplus_{i=1}^{n} Ae_i$.

Now, suppose that $e_i$ is a primitive idempotent, but that $Ae_i = B_i \oplus C_i$ for nonzero left ideals $B_i$ and $C_i$ of $Ae_i$. Then there exist unique elements $f_i \in B_i$ and $g_i \in C_i$ such that $e_i = f_i + g_i$. Since $f_i \in Ae_i$, we have that $f_i e_i = f_i$. Hence,

$$f_i = f_i^2 = f_i(e_i - g_i) = f_i - f_i g_i,$$

so that $f_i g_i = 0$. Similarly, $g_i f_i = 0$. Thus,

$$f_i + g_i = (f_i + g_i)^2 = f_i^2 + g_i^2,$$

so that $g_i$ and $f_i$ are mutually orthogonal idempotents whose sum is $e_i$. Since $e_i$ is primitive, this implies that one of $f_i$ and $g_i$ is 0. Without loss, we may assume $f_i = 0$. Then $g_i = e_i$, and $C_i$ is a left ideal of $Ae_i$ which contains $g_i = e_i$. This implies that $C_i = Ae_i$ which yields that $B_i = 0$, a contradiction. □

Combining Lemmas 2.1 and 2.2, we see that if $I$ is a left ideal of $A$, then $I$ is a principal indecomposable module of $A$ if and only if $I = Ae$ for some primitive idempotent $e \in A$. We will be particularly interested in the $A$-endomorphism algebras of such modules.

**Lemma 2.3.** *Let $A$ be an algebra over a field and $e \in A$ an idempotent. Then $Ae$ is a left $A$-module and*

$$\mathrm{End}_A(Ae) \simeq (eAe)^{\mathrm{op}}$$

*as $F$-algebras.*

*Proof.* Let $E = \mathrm{End}_A(Ae)$. First, note that every $f \in E$ is completely determined by $f(e)$ since $f(ae) = af(e)$ for all $a \in A$. Now, define a map

$$\Phi : E \to (eAe)^{\mathrm{op}}$$

by $\Phi(f) = ef(e)e$ for all $f \in E$. Then, observe that if $b \in Ae$, then $b = ae$ for some $a \in A$, and hence, $be = aee = ae^2 = ae = b$. Hence, since $e \in A$ and $f(e) \in Ae$, we have that

$$\Phi(f) = ef(e)e = f(e^2)e = f(e)e = f(e) \in eAe$$

for all $f \in E$. It is clear that $\Phi$ is a vector space homomorphism. Now, let $f, g \in E$. Then

$$\Phi(f \circ g) = f(g(e)),$$

and

$$\Phi(f) \cdot_{\mathrm{op}} \Phi(g) = \Phi(g)\Phi(f) = g(e)f(e) = f(g(e))$$

since $g(e) \in A$ and $f$ is an $A$-endomorphism. Also, since $\Phi(\mathrm{Id}_{Ae}) = e$, which is the identity of $eAe$, it follows that $\Phi$ is an $F$-algebra homomorphism. Now, if $f \in \ker(\Phi)$, then $0 = f(e) = af(e) = f(ae)$ for all $a \in A$, which implies that $f$ is the zero map on $Ae$, so $\Phi$ is injective.

Now, we will show that $\Phi$ is surjective. Let $x \in eAe$. Then $x = ewe$ for some $w \in A$. Hence, we may define a map

$$\sigma_x : Ae \to Ae$$

by $\sigma_x(s) = sx \in Ae$ for all $s \in Ae$. We will show that $\sigma_x \in E$. Let $\gamma \in F, s, t \in Ae$. Then

$$\sigma_x(\gamma s + t) = (\gamma s + t)x = \gamma sx + tx = \gamma\sigma_x(s) + \sigma_x(t),$$

so $\sigma_x$ is a vector space homomorphism. Furthermore, if $a \in A$, then

$$\sigma_x(as) = asx = a\sigma_x(s),$$

so $\sigma_x \in E$. Now,

$$\Phi(\sigma_x) = \sigma_x(e) = ex = eewe = ewe = x.$$

Hence, $\Phi$ is surjective and the proof is complete. $\square$

**Corollary 2.4.** *Let ${}_AA$ be the left $A$-module $A$. Then*

$$A \simeq \mathrm{End}_A({}_AA)^{\mathrm{op}}.$$

*as $F$-algebras.*

*Proof.* Set $e = 1$ and apply Lemma 2.3. $\square$

Now, recall the definition of basic algebra of $A$ as in [1, p. 23]. Let ${}_AA$ be the regular left $A$-module $A$. We may write

$${}_AA = \bigoplus_{i=1}^{n} A_i,$$

where $A_1, \dots, A_n$ are principal indecomposable modules of $A$. It may happen that some of the $A_i$ are isomorphic. Reorder the $A_i$ if necessary so that $A_1, \dots A_r$, $r \leq n$, is a complete set of nonisomorphic principal indecomposable modules. Then the $F$-algebra $\mathrm{End}_A(A_1 \oplus \cdots \oplus A_r)^{\mathrm{op}}$ is the *basic algebra* of $A$. Note that the basic algebra is defined up to isomorphism, so we may, at times, refer to *a* basic algebra of $A$.

Two algebras are said to be *Morita equivalent* if they have isomorphic basic algebras. A theorem of Morita in [16] yields that the basic algebra of $A$ is an invariant of the category of $A$-modules.

**Theorem 2.5** (Morita)**.** *Let $A$ and $B$ be two $F$-algebras. Let ${}_A\mathbf{Mod}$ and ${}_B\mathbf{Mod}$ be the categories of left $A$- and $B$-modules, respectively. Then ${}_A\mathbf{Mod}$ and ${}_B\mathbf{Mod}$ are equivalent as abelian categories if and only if $A$ and $B$ are Morita equivalent. Furthermore, $A$ and* $\mathrm{Mat}(n, A)$ *are Morita equivalent.* $\square$

A well-known feature of the basic algebra is that all of its irreducible representations are linear. We prove this fact by considering idempotents below.

**Theorem 2.6.** *Let $A$ be a finite dimensional $F$-algebra. Then the basic algebra of $A$ has the same number of irreducible representations as $A$, and every irreducible representation of the basic algebra of $A$ is linear.*

*Proof.* Let $A_1, \dots, A_r$ be a complete set of nonisomorphic principal indecomposable $A$-modules. By [6, Corollary (15.13)] and [6, Corollary (15.14)], $A_i \mapsto \mathrm{Top}(A_i)$ is a bijection from $\{A_1, \dots, A_r\}$ to the set of isomorphism classes of irreducible representations of $A$. In particular, the number of irreducible representations of $A$ is $r$. Now, since $A_1, \dots, A_r$ are PIMs, there exists a set of primitive orthogonal idempotents $e_1, \dots, e_r$ such that $A_i = Ae_i$ for all $i$. Hence, if we set $e = e_1 + \cdots + e_r$, then $E = eAe$ is a basic algebra of $A$. It suffices to show that the semisimple algebra $E/J(E)$ is isomorphic to a direct sum of $r$ copies of $F$.

Note that $\{e_1 = ee_1e, \dots, e_r = ee_re\} \subseteq E$ is a set of primitive orthogonal idempotents of $E$ whose sum is $e$, the identity of $E$. Hence, by Lemma 2.2, $E = Ee_1 \oplus \cdots \oplus Ee_r$ as a direct sum of primitive indecomposable modules. But, $Ee_i = eAee_i = eAe_i$. Since $\{e_1, \dots, e_r\}$ is a set of orthogonal idempotents, we have that $Ee_i = e_1Ae_i \oplus \cdots \oplus e_rAe_i$ as a vector space for each $1 \leq i \leq r$.

Now, we claim that if $i \neq j$, then $e_iAe_j \subseteq J(E)$. Indeed, since $e_ie_j = 0$, the cosets $e_i + J(A)$ and $e_j + J(A)$ live in two different minimal two-sided ideals, say $I$ and $J$, respectively, of the semisimple algebra $A/J(A)$. But then $e_iAe_j + J(A) \subseteq I \cap J$, so that $e_iAe_j \subseteq J(A)$. Now, by [6, Theorem (54.6)], $J(E) = eJ(A)e$. Hence, since $e_iAe_j \subseteq J(A)$, so that $e_iAe_j = ee_iAe_je \subseteq J(E)$.

Now, since $E = \sum_{i,j=1}^{r} e_iAe_j$, and $J(E) = eJ(A)e$ is an $F$-algebra in its own right, we have that

$$E = (e_1Ae_1 \oplus \cdots \oplus e_rAe_r) + J(E)$$

as a sum of $F$-algebras. Now, we claim that $e_iAe_i \cap J(E) = e_iJ(A)e_i$. Indeed, if $e_ixe_i \in e_iJ(A)e_i$ for $x \in J(A) \subseteq A$, then $e_ixe_i = ee_ixe_ie \in eJ(A)e = J(E)$, so that $e_iJ(A)e_i \subseteq e_iAe_i \cap J(E)$. If $y \in e_iAe_i \cap J(E)$, then $y = e_ixe_i = eze$ for some $x \in A$, $z \in J(A)$, then by multiplying on both sides by $e_i$, we get $y = e_ixe_i = e_ize_i \in e_iJ(A)e_i$, so that $e_iAe_i \cap J(E) = e_iJ(A)e_i$.

Thus,

$$\begin{aligned} E/J(E) &= ((e_1Ae_1 \oplus \cdots \oplus e_rAe_r) + J(E))/J(E) \\ &\simeq e_1Ae_1/e_1J(A)e_1 \oplus \cdots \oplus e_rAe_r/e_rJ(A)e_r. \end{aligned}$$

Now, by [6, Theorem (54.9)] and [6, Lemma (54.8)], the algebra

$e_iAe_i/J(e_iAe_i)$ is a skewfield, but as $F$ is an algebraically closed field, this implies that $e_iAe_i/J(e_iAe_i) \simeq F$ as an $F$-algebra. Finally, by [6, Theorem (54.6)] again, $J(e_iAe_i) = e_iJ(A)e_i$, and hence, by the above, $E/J(E)$ is isomorphic to the direct sum of $r$ copies of $F$ as an $F$-algebra, completing the proof of the theorem. □

**Theorem 2.7.** *Suppose $A$ is a finite dimensional $F$-algebra and that every irreducible representation of $A$ is one-dimensional. Then $A$ is isomorphic to its basic algebra.*

*Proof.* Since every irreducible representation of $A$ is one-dimensional, the semisimple algebra $A/J(A)$ is isomorphic to $F^{\oplus r}$, where $r$ is the number of irreducible representations of $A$. Let $e_i$ be the identity of the $i$-th direct summand of $F^{\oplus r}$. Then $e_i$ is primitive, $e_ie_j = 0$ if $i \neq j$, and $e_1 + \cdots + e_r$ is the identity of $A/J(A)$. Hence, by [1, Corollary 1.7.4], there exist primitive orthogonal idempotents $f_1, \ldots, f_r \in A$ such that $1 = f_1 + \cdots + f_r$. Hence, by Lemma 2.2, $A = \bigoplus_{i=1}^{r} Af_i$. By [6, Corollary (54.13)] and [6, Corollary (54.13)], the map $Af_i \mapsto \mathrm{Top}(Af_i)$ is a surjection from the set of isomorphism classes of principal indecomposable $A$-modules to the set of isomorphism classes of irreducible representations. Since $A$ has $r$ irreducible representations, this map is a bijection. Hence, $Af_i$ and $Af_j$ are nonisomorphic if $i \neq j$.

Finally, the basic algebra of $A$ is

$$\operatorname{End}_A\left(\bigoplus_{i=1}^{r} Af_i\right)^{\mathrm{op}} = \operatorname{End}_A(A)^{\mathrm{op}} \simeq A$$

by Corollary 2.4. $\square$

Let $\{A_1, \ldots, A_r\}$ be a complete set of nonisomorphic principal indecomposable $A$-modules. Then, by Lemma 2.1, $A_i = Ae_i$ for some primitive orthogonal idempotents $e_1, \ldots, e_r \in A$. Now, note that since the $e_i$ are orthogonal idempotents, we have that

$$\bigoplus_{i=1}^{n} A_i = \bigoplus_{i=1}^{r} Ae_i = A(e_1 + \cdots + e_r).$$

Hence, by Lemma 2.3, the basic algebra of $A$ is isomorphic to $eAe$, where $e = e_1+\cdots+e_r$. Note that since the $e_i$ are orthogonal, $e$ is an idempotent. An idempotent $e \in A$ of this form is called a *basic idempotent* of $A$.

Theorems 2.6 and 2.7 imply that $A$ is isomorphic to its basic algebra if and only if every irreducible representation of $A$ is linear. In this case, we say that the algebra $A$ is *basic*.

**Theorem 2.8.** *Let $A$ be a finite dimensional $F$-algebra, and let $e \in A$ be a basic idempotent. Then $A$ is basic if and only if $e = 1$, the identity of $A$.*

*Proof.* We have that $eAe \subseteq A$. Also, $eAe$ is an algebra with identity $e$. If $A$ is basic, then $eAe$ and $A$ have the same dimension as $F$-vector spaces. In particular, $eAe = A$. Since $A$ is an algebra with identity 1, it must be that $e = 1$. Conversely, if $e = 1$, then $eAe = A$. $\square$

**Theorem 2.9.** *Suppose $A$ is an algebra over an algebraically closed field and that $A$ has a unique (up to isomorphism) irreducible representation. Then $A$ is isomorphic to the algebra of $n$-by-$n$ matrices over the basic algebra of $A$, where $n$ is the degree of the unique irreducible representation of $A$.*

*Proof.* Let $\bar{A}$ be the left $A$-module $A$. Then, since $A$ has a unique irreducible representation, we may write

$$\bar{A} \simeq P \oplus \ldots \oplus P = mP$$

where $P$ is a projective cover of the unique irreducible $A$-module. By Corollary 2.4, $A \simeq \mathrm{End}_A(\bar{A})^{\mathrm{op}}$. Hence,

$$A \simeq \mathrm{End}_A(\bar{A})^{\mathrm{op}} \simeq \mathrm{End}_A(mP)^{\mathrm{op}} \simeq (\mathrm{End}_A(P) \otimes_F \mathrm{Mat}(m, F))^{\mathrm{op}} \simeq \mathrm{End}_A(P)^{\mathrm{op}} \otimes_F \mathrm{Mat}(m, F).$$

Hence, since $\mathrm{End}_A(P)^{\mathrm{op}}$ is the basic algebra of $A$, $A$ is isomorphic to $m$-by-$m$ matrices over its basic algebra. Since the unique irreducible representation of $A$ can be viewed as an irreducible representation of the basic algebra (all of which are linear by 2.6) tensored with an irreducible representation of $\mathrm{Mat}(m, F)$ (which has degree $m$), we have that $m = n$ and we are done. □

### 2.2 Characters and Blocks of Finite Groups

In this section, we recall some basic facts about characters and blocks of $p$-solvable groups. Throughout, let $G$ be a finite group. Fix a prime number $p$.

We first recall the definition of an ordinary character of $G$. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, $n \geq 0$, be a $\mathbb{C}$-representation (which is also called an ordinary representation) of $G$. Recall that the *character* $\chi$ afforded by $\rho$ is the map $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{Tr}(\rho(g))$, where Tr is the trace function. The *degree* of $\chi$ is the integer $n$. We say that a character $\chi$ is *irreducible* if the representation which affords $\chi$ is irreducible. We write $\mathrm{Irr}(G)$ for the set of irreducible characters of $G$. A standard text on the character theory of finite groups is [12].

By Maschke's Theorem (see, for example, [17, Theorem (1.21)]), the group algebra $\mathbb{C}G$ is semisimple and is, hence, by Wedderburn's Theorem [17, Theorem (1.17)], $\mathbb{C}G$ is a direct sum of full matrix algebras. Each character $\chi \in \mathrm{Irr}(G)$ corresponds uniquely to one of these full matrix algebras in the following way. For $\chi \in \mathrm{Irr}(G)$, let

$$f_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} g \in Z(\mathbb{C}G),$$

the primitive idempotent of $Z(\mathbb{C}G)$ associated to $\chi$. Then $\mathbb{C}Gf_\chi$ is a full matrix algebra over $\mathbb{C}$ of degree $\chi(1)$, and as a $\mathbb{C}G$-module, $\mathbb{C}Gf_\chi$ is isomorphic to the direct sum of $\chi(1)$ copies of $M$, where $M$ is the irreducible $\mathbb{C}G$-module which affords $\chi$.

We will now recall the definition of a Brauer (or modular) character of $G$, which are the characteristic $p$ analogs of ordinary characters, as in [17]. First, we will define a particular algebraically closed field of characteristic $p$. Let $R$ be the ring of algebraic integers in $\mathbb{C}$. Choose a maximal ideal $M$ of $R$ which contains the integer $p$. Then $R/M$ is a field of characteristic $p$. Let $F = R/M$. Define $^* : R \to F$ to be the natural surjective ring homomorphism given by $r^* = r + M$ for all $r \in R$. We call $^*$ *reduction modulo* $p$. The following lemma shows that $F$ is the appropriate field to define Brauer characters.

**Lemma 2.10.** *Let $U \in \mathbb{C}$ be the group of $p'$-roots of unity. That is, $U$ is the multiplicative group of complex roots of* 1 *whose orders are not divisible by $p$. Then the restriction of $^*$ to $U$ defines an isomorphism $^* : U \to F^\times$ of multiplicative groups. Furthermore, $F$ is the algebraic closure of its prime field $\mathbb{Z}^* \simeq \mathbb{Z}_p$.*

Let $G^0$ be the set of $p$-regular elements of $G$. That is, $G^0$ is the set of elements of $G$ whose orders are not divisible by $p$. Let $\rho : G \to \mathrm{GL}_n(F)$ be an $F$-representation of $G$. If $g \in G^0$, then (counting multiplicity) $\rho(g)$ has $n$ eigenvalues, all of which are in $F^\times$. Hence, by Lemma 2.10, the eigenvalues of $\rho(g)$ can be written as $\xi_1^*, \ldots, \xi_n^* \in F^\times$ for uniquely determined $\xi_1, \ldots, \xi_n \in U$. We now say that the *Brauer character* (or *modular character*) afforded by $\rho$ is the map $\varphi : G^0 \to \mathbb{C}$ defined by $\varphi(g) = \xi_1 + \cdots + \xi_n$ as above. We say that a Brauer character $\varphi$ is *irreducible* it is afforded by an irreducible representation. Denote the set of irreducible Brauer characters by $\mathrm{IBr}_p(G)$. Throughout, since the prime $p$ is fixed, we will simply write $\mathrm{IBr}(G)$.

Block theory (for which a standard text is [17]) is a tool for relating the ordinary and Brauer characters of a finite group. Since these characters carry information about their corresponding representations, block theory is also used to study the relationships between the ordinary and modular representations of $G$ (equivalently, $\mathbb{C}G$-modules and $FG$-modules).

We will define the $p$-blocks of $G$ as in [17]. For $\chi \in \mathrm{Irr}(G)$, let $\chi^0 = \chi|_{G^0}$, the restriction of $\chi$ to $G^0$. By [17, Corollary (2.9)], $\chi^0$ is a Brauer character of $G$. Hence, there exist integers $d_{\chi\varphi} \geq 0$ indexed by $\varphi \in \mathrm{IBr}(G)$ such that

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi.$$

The integers $d_{\chi\varphi}$, $\chi \in \mathrm{Irr}(G)$, $\varphi \in \mathrm{IBr}(G)$ are called the *decomposition numbers*. We can now define the blocks of $G$.

**Definition 2.11.** We say that two ordinary characters $\chi, \psi \in \mathrm{Irr}(G)$ are *linked* if there exists some $\varphi \in \mathrm{IBr}(G)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}$. The *Brauer graph* of $G$ is the graph whose vertices are the elements of $\mathrm{Irr}(G)$ and adjacency is defined by linkage. A *block* $B$ of $G$ is a subset of $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ containing subsets $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G)$ and $\mathrm{IBr}(B) \subseteq \mathrm{IBr}(G)$ such that:

- $\mathrm{Irr}(B)$ is a connected component of the Brauer graph.
- $\mathrm{IBr}(B) = \{\varphi \in \mathrm{IBr}(G) \mid d_{\chi\varphi} > 0 \text{ for some } \chi \in \mathrm{Irr}(B)\}$.
- $B = \mathrm{Irr}(B) \cup \mathrm{IBr}(B)$.

We denote the set of blocks of $G$ by $\mathrm{Bl}(G)$.

Each block of $G$ corresponds to an indecomposable two-sided ideal of the group algebra $FG$. If $B \in \mathrm{Bl}(G)$, then the *block idempotent* of $B$, $e_B$ is given by

$$e_B = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi,$$

where

$$e_\chi = f_\chi^* = \left( \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} g \right)^* \in Z(FG).$$

Then each $e_B$ is a primitive idempotent of $Z(FG)$, $FGe_B$ is an indecomposable two-sided ideal of $FG$, and $FG = \sum_{B \in \mathrm{Bl}(G)} FGe_B$. Given $B \in \mathrm{Bl}(G)$, $FGe_B$ is the *block algebra* of $B$.

Clifford theory is the study of the relationships between the representation theory of a group and a normal subgroup. Such relationships were originally studied by Alfred Clifford in [5]. If $N$ is a normal subgroup of $G$, then there is an action of $G$ on $\mathrm{Irr}(N)$ (respectively $\mathrm{IBr}(N)$) given by $\theta^g(x) = \theta(gxg^{-1})$ for all $\theta \in \mathrm{Irr}(N)$ (respectively $\theta \in \mathrm{IBr}(N)$). If $\theta \in \mathrm{Irr}(N) \cup \mathrm{IBr}(N)$, we denote by $I_G(\theta)$ the stabilizer in $G$ of this action on $\theta$. We call $I_G(\theta)$ the *inertia group* of $\theta$ in $G$. We will now summarize some results from Clifford theory. We will first state the ordinary and modular versions of Clifford's theorem and Clifford correspondence. The ordinary versions are in [12, Theorem (6.2)] and [12, Theorem (6.11)], repectively. The modular versions are in [17, Corollary (8.7)] and [17, Theorem (8.9)], respectively.

**Theorem 2.12** (Clifford's Theorem, Ordinary Version)**.** *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$ and let $\chi \in \mathrm{Irr}(G)$. Let $\theta$ be an irreducible constituent of $\chi_N$ and suppose $\theta = \theta_1, \dots, \theta_t$ are the distinct conjugates of $\theta$ in G. Then*

$$\chi_N = e \sum_{i=1}^{t} \theta_i$$

*where $e = [\chi_N, \theta]$.*

**Theorem 2.13** (Clifford Correspondence, Ordinary Version)**.** *Let $N \trianglelefteq G$, let $\theta \in \mathrm{Irr}(N)$, and $T = I_G(\theta)$. Then $\psi \mapsto \psi^G$ is a bijection from $\mathrm{Irr}(T \mid \theta)$ onto $\mathrm{Irr}(G \mid \theta)$.*

**Theorem 2.14** (Clifford's Theorem, Modular Version)**.** *Let $N \trianglelefteq G$. Let $\varphi \in \mathrm{IBr}(G)$ and let $\theta \in \mathrm{IBr}(N)$. Then $\varphi$ is an irreducible constituent of $\theta^G$ if and only if $\theta$ is an irreducible* *constituent of $\varphi_N$. In this case, if $\theta = \theta_1, \dots, \theta_t$ are the distinct G-conjugates of $\theta$, then*

$$\varphi_N = e \sum_{i=1}^{t} \theta_i.$$

**Theorem 2.15** (Clifford Correspondence, Modular Version)**.** *Let $N \trianglelefteq G$, let $\theta \in \mathrm{IBr}(N)$, and $T = I_G(\theta)$. Then $\psi \mapsto \psi^G$ is a bijection from $\mathrm{IBr}(T \mid \theta)$ onto $\mathrm{IBr}(G \mid \theta)$.*

We can refine the result of Theorem 2.14 in the case that $G/N$ is a $p$-group. This is the content of Green's Theorem [17, Theorem (8.11)].

**Theorem 2.16** (Green)**.** *Suppose that $G/N$ is a p-group and let $\theta \in \mathrm{IBr}(N)$. Then there exists a unique $\varphi \in \mathrm{IBr}(G \mid \theta)$. Furthermore, $\varphi_N$ is the sum of the distinct G-conjugates of $\theta$. In particular, if $\theta$ is G-invariant, then $\varphi_N = \theta$.*

The action of $G$ on $\mathrm{Irr}(N)$ and $\mathrm{IBr}(N)$ also preserves the block strucure of $G$. More precisely, if $b \in \mathrm{Bl}(N)$, then the set $b^g = \{\theta^g \mid \chi \in \mathrm{Irr}(b) \cup \mathrm{IBr}(b)\}$ is also a block of $G$. Thus, $G$ acts on $\mathrm{Bl}(N)$ by conjugation.

We wish to state a Clifford theorem for blocks, but first we must discuss block coverings. Given $B \in \mathrm{Bl}(G)$ and $b \in \mathrm{Bl}(N)$, we say that *$B$ covers $b$* if there exist blocks $B = B_1, \dots, B_s \in \mathrm{Bl}(G)$ and $b = b_1, \dots, b_t \in \mathrm{Bl}(N)$ such that

$$\sum_{i=1}^{s} e_{B_i} = \sum_{i=1}^{t} e_{b_s}.$$

An equivalent definition of block covering can be found in [17, Theorem (9.2)], which we state now.

**Theorem 2.17.** *Let $N$ be a normal subgroup of $G$. Let $b \in \mathrm{Bl}(N)$ and let $B \in \mathrm{Bl}(G)$. Then the following are equivalent:*

- *$B$ covers $b$.*
- *If $\chi \in B$, then every irreducible constituent of $\chi_N$ lies in a G-conjugate of $b$.*
- *There is a $\chi \in B$ such that $\chi_N$ has an irreducible constituent in $b$.*

Note that Theorem 2.17 implies that if $b_1, b_2$ are both covered by $B$, then $b_1$ and $b_2$ are $G$-conjugate. We denote the set of blocks which cover $b$ by $\mathrm{Bl}(G \mid b)$.

We conclude this section with the Fong-Reynolds Theorem [17, Theorem (9.14)], which is an analog of the Clifford theorems for blocks.

**Theorem 2.18.** *Let $N$ be a normal subgroup of $G$ and let $b \in \mathrm{Bl}(N)$. Let $T(b)$ be the stabilizer of $b$ in $G$.*

- *The map $\mathrm{Bl}(T(b) \mid b)$ to $\mathrm{Bl}(G \mid b)$ given by $B \mapsto B^G$ is a well-defined bijection.*
- *If $B \in \mathrm{Bl}(T(b) \mid b)$, then $\mathrm{Irr}(B^G) = \{\psi^G \mid \psi \in \mathrm{Irr}(B)\}$ and $\mathrm{IBr}(B^G) = \{\varphi^G \mid \varphi \in \mathrm{IBr}(B)\}$.*
- *If $B \in \mathrm{Bl}(T(b) \mid b)$, then every defect group of $B$ is a defect group of $B^G$.*

## 2.3 Characters and Blocks of $p$-Solvable Groups

Let $G$ be a finite group, and let $\pi$ be a set of primes. A Hall $\pi$-subgroup of $G$ is a subgroup $H$ of $G$ such that $|H|$ is divisible only by primes in $\pi$ and $|G : H|$ is not divisible by any primes in $\pi$. Hence, for $q$ a prime, the Hall $\{q\}$-subgroups of $G$ are precisely the Sylow $q$-subgroups. Recall that $G$ is *solvable* if every composition factor of $G$ is cyclic of prime order. Phillip Hall's theorem, which appears in [13, Theorem 6.4.10] characterizes solvable groups in terms of Hall subgroups.

**Theorem 2.19** (P. Hall)**.** *Let $G$ be a finite group. Then the following are equivalent:*

- *$G$ is solvable.*
- *$G$ contains a Hall $\pi$-subgroup for every set of primes $\pi$.*
- *$G$ contains a Hall $p'$-subgroup for every prime $p$.*

A Hall $p'$-subgroup is also called a *$p$-complement*. Recall that a group $G$ is $p$-solvable if every nonabelian composition factor of $G$ is a $p'$-group. From [13, Theorem 6.4.6], it follows that a $p$-solvable group has a $p$-complement and that every $p$-complement is conjugate in $G$.

We will now discuss the characters and blocks of $p$-solvable groups. Recall that a finite group $G$ is $p$-solvable if every nonabelian composition factor of $G$ is a $p'$-group.

The characters and blocks of $p$-solvable groups were originally studied by Paul Fong in [7]. The main results in the character theory and block theory of $p$-solvable groups (which are surveyed in [17, Chapter 10]) reduce the modular representation theory to ordinary representation theory in some way. The first such result we will list is the famed Fong-Swan theorem, which appears in [17, Theorem (10.1)].

**Theorem 2.20** (Fong-Swan)**.** *Suppose that $G$ is $p$-solvable. If $\varphi \in \mathrm{IBr}(G)$, then $\varphi = \chi^0$ for some $\chi \in \mathrm{Irr}(G)$.*

The Fong-Swan theorem allows us to calculate the irreducible Brauer characters of a $p$-solvable group completely in terms of the ordinary characters, as demonstrated in the following corollary, found in [17, Corollary (10.4)].

**Corollary 2.21.** *Suppose $G$ is $p$-solvable. Then $\mathrm{IBr}(G)$ is the set of all $\chi^0$ such that $\chi \in \mathrm{Irr}(G)$ and $\chi^0$ is not of the form $\alpha^0 + \beta^0$ for nonzero ordinary characters $\alpha, \beta$ of $G$.*

Let $\varphi \in \mathrm{IBr}(G)$, where $G$ is $p$-solvable. A *lift* of $\varphi$ is a character $\chi \in \mathrm{Irr}(G)$ such that $\varphi = \chi^0$. There may be many lifts of a given irreducible Brauer character $\varphi$. A theorem of Isaacs, which appears in [17, Theorem (10.6)], states that (for $p$ odd) there is subset of $\mathrm{Irr}(G)$ such that lifting to this subset is injective. Recall that a character $\chi \in \mathrm{Irr}(G)$ is *$p$-rational* if its values are in $\mathbb{Q}(|G|_{p'})$, the field obtained by adjoining to $\mathbb{Q}$ a primitive $p'$-root of unity.

**Theorem 2.22** (Isaacs)**.** *Suppose that $G$ is $p$-solvable, and $p$ is odd. If $\varphi \in \mathrm{IBr}(G)$, then there exists a unique $p$-rational lift of $\varphi$.*

Since $p$-solvable groups have $p$-complements, we can discuss the relationships between the characters of a $p$-solvable group $G$ and those of a $p$-complement $H$ of $G$. Recall that since $p$ does not divide the order of $H$, $\mathrm{IBr}(H) = \mathrm{Irr}(H)$. The following theorem, found in [17, Theorem (10.9)], shows what happens when irreducible Brauer characters of $G$ are restricted to $H$.

**Theorem 2.23.** *Suppose that $G$ is $p$-solvable and let $H$ be a $p$-complement of $G$. Then the map $\varphi \mapsto \varphi_H$ from $\{\varphi \in \mathrm{IBr}(G) \mid p \nmid \varphi(1)\} \to \mathrm{Irr}(H)$ is a well-defined injective map.*

Let $M$ be an irreducible $FG$-module with projective cover $P$. If $M$ affords $\varphi \in \mathrm{IBr}(G)$, then recall that $P$ affords the Brauer character

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi^0.$$

The Brauer character $\Phi_\varphi$ is called the *projective indecomposable character* of $\varphi$. A theorem of Fong in [17, Theorem (10.13)] states that the projective indecomposable characters are induced from certain irreducible characters of $p$-complement subgroups.

**Theorem 2.24** (Fong)**.** *Suppose that $G$ is $p$-solvable and let $H$ be a $p$-complement of $G$. If $\varphi \in \mathrm{IBr}(G)$, then there exists a character $\alpha$ of $H$ such that $\alpha^G = \Phi_\varphi$. Furthermore, every such $\alpha$ is irreducible and has degree $\varphi(1)_{p'}$.*

**Corollary 2.25** (Fong's Dimension Formula)**.** *If $G$ is $p$-solvable and $\varphi \in \mathrm{IBr}(G)$, then*

$$\Phi_\varphi(1) = |G|_p \varphi(1)_{p'}.$$

A character $\alpha$ of a $p$-complement $H$ such that $\alpha^G = \Phi_\varphi$ is said to be a *Fong character* for $\varphi$. The next theorem, [17, Theorem (10.18)], identifies the Fong characters.

**Theorem 2.26.** *Suppose $G$ is $p$-solvable and $H$ is a $p$-complement of $G$. If $\varphi \in \mathrm{IBr}(G)$, then the Fong characters of $\varphi$ are the irreducible constituents of $\varphi_H$ of smallest possible degree. This degree is $\varphi(1)_{p'}$.*

To conclude this section, we state a theorem of Fong, found in [17, Theorem (10.20)], which, together which the theory of block covering and Theorem 2.18 is the main tool for studying blocks of $p$-solvable groups.

**Theorem 2.27** (Fong)**.** *Suppose $G$ is $p$-solvable and let $\theta \in \mathrm{Irr}(O_{p'}(G))$ be $G$-invariant. Then there is a unique $p$-block of $G$ covering the block $\{\theta\}$ of $O_{p'}(G)$. Also, $\mathrm{Irr}(B) = \mathrm{Irr}(G \mid \theta)$ and $\mathrm{IBr}(B) = \mathrm{IBr}(G \mid \theta)$. Furthermore, the defect groups of $B$ are the Sylow $p$-subgroups of $G$.*

Finally, we state a corollary to Theorem 2.27, as found in [11], which shows that the hypothesis that $\theta$ is $G$-invariant is not too restrictive.

**Corollary 2.28** (Isaacs)**.** *Let $G$ be $p$-solvable, and let $B \in \mathrm{Bl}(G)$. Then there exists a subgroup $J$ of $G$ and a character $\psi \in \mathrm{Irr}(M)$, where $M = O_{p'}(J)$, such that induction defines bijections from $\mathrm{Irr}(J \mid \psi)$ onto $\mathrm{Irr}(B)$ and from $\mathrm{IBr}(J \mid \psi)$ onto $\mathrm{IBr}(B)$.*

# CHAPTER 3
# BLOCK IDEMPOTENTS FOR *P*-SOLVABLE GROUPS

In this section, we discuss the block idempotents $e_B$ for blocks $B$ of $p$-solvable groups. We proceed using the notation of Chapter 2.

## 3.1 Blocks of Full Defect

If $B$ is a block of a finite group $G$, then $B$ of said to be *of full defect* (or *has full defect* if the defect groups of $B$ are precisely the Sylow $p$-subgroups of $G$. Recall that $B$ has full defect if and only if $B$ contains an irreducible ordinary character whose degree is not divisible by $p$. By [17, Corollary (3.17)], another equivalent condition for $B$ to have full defect is that $B$ contains an irreducible Brauer character whose degree is not divisible by $p$.

To begin, we provide an English translation of a theorem which appears in German in [19, Satz 2.1].

**Theorem 3.1** (Tiedt). *Let $B$ be a block of a $p$-solvable group $G$. If a defect group of $B$ contains a Sylow $p$-subgroup of $O_{p'p}(G)$, then the support of $e_B$ is contained in $O_{p'}(G)$.*

The proof of Theorem 3.1 relies on several deeper results, due to Knörr, Harris, and Osima, presented here as lemmas. These results can be found in [17, Theorem (9.26)], [9, Corollary 2], and [17, Corollary (3.8)], respectively.

**Lemma 3.2** (Knörr's Theorem). *Let $G$ be a finite group. Let $b \in \mathrm{Bl}(N|Q)$, where $N \lhd G$. If $B \in \mathrm{Bl}(G)$ covers $b$, then there is a defect group $P$ of $B$ such that $P \cap N = Q$.*

**Lemma 3.3** (Harris-Knörr). *Let $G$ be a finite group. Let $N \lhd G$ and $b \in \mathrm{Bl}(N|Q)$. If $C_G(Q) \leq N$, then $b^G$ is defined and is the only block of $G$ which covers $b$.*

**Lemma 3.4** (Osima's Theorem). *Let $G$ be a finite group. If $B \in \mathrm{Bl}(G)$, then $e_B$ has support in $G^0$.*

We can now prove Theorem 3.1.

*Proof of Theorem 3.1.* Suppose $G$ is a minimal counterexample. Suppose $G = O_{p'p}(G)$. Then $G$ is $p$-nilpotent and hence $G^0 = O_{p'}(G)$. Then, by Lemma 3.4, the support of

$e_B$ is contained in $O_{p'}(G)$, a contradiction. Hence, $O_{p'p}(G)$ is a proper subgroup of $G$. Consider the ascending $p$-series

$$1 \leq O_{p'}(G) \leq O_{p'p}(G) \leq O_{p'pp'}(G) \leq \cdots \leq G.$$

Let $H$ be the largest proper subgoup of $G$ in the above series. Then $H \triangleleft G$ and $O_{p'p}(G) \leq H$. Since $H$ is normal in $G$ and $O_{p'}(G) \leq H$, $O_{p'}(H) = O_{p'}(G)$. Similarly, $O_{p'p}(H) = O_{p'p}(G)$.

Now, by [17, Corollary (9.3)], $B$ covers exactly one $G$-conjugacy class of blocks $\{b_1, \dots, b_n\}$ of $H$. Then, by definition of block covering,

$$\sum_{i=1}^{n'} e_{B_i} = \sum_{j=1}^{n} e_{b_j},$$

where $\{B = B_1, B_2, \dots, B_{n'}\}$ is the collection of blocks of $G$ which cover $\{b_1, \dots, b_n\}$. It suffices to show that $n' = 1$ and that each $e_{b_j}$ has support in $O_{p'}(H)$.

For each $1 \leq j \leq n$, let $Q_j$ be a defect group of $b_j$. Then, by Lemma 3.2, there exist defect groups $P_j$ of $B$ such that $Q_j = P_j \cap H$ for all $j$. Let $S_j$ be a Sylow $p$-subgroup of $O_{p'p}(G)$ such that $S_j \leq P_j$. Then, since $S_j \leq O_{p'p}(G) \leq H$ and $S_j \leq P_j$, it holds that $S_j \leq Q_j = P_j \cap H$. Now, since $Q_j$ contains a Sylow subgroup of $O_{p'p}(G)$ and $O_{p'p}(H) = O_{p'p}(G)$, $Q_j$ contains a Sylow subgroup of $O_{p'p}(H)$. By induction, each $e_{b_j}$ has support in $O_{p'}(H)$.

Finally, since $G$ is $p$-constrained, we have that $C_G(S_1) \leq O_{p'p}(G)$. Hence,

$$C_G(Q_1) \leq C_G(S_1) \leq O_{p'p}(G) \leq H.$$

Hence, by Lemma 3.3, $B$ is the only block of $G$ that covers $b_1$. Thus,

$$e_B = \sum_{j=1}^{n} e_{b_j}$$

and we are done. $\square$

**Corollary 3.5** (Tiedt)**.** *Let $B$ be a block of a $p$-solvable group $G$. If $B$ has full defect, then the support of $e_B$ is contained in $O_{p'}(G)$.*

*Proof.* Since $B$ has full defect, the defect groups of $B$ are precisely the Sylow $p$-subgroups of $G$. Hence, any Sylow subgroup of $O_{p'p}(G)$ is contained in some defect group of $B$, and we may apply Theorem 3.1. $\square$

We will now provide some alternate ways of calculating the block idempotent $e_B$ of a block $B$ of a $p$-solvable group. Theorem 3.6 provides a calculation of $e_B$ in terms of irreducible characters of $O_{p'}(G)$ and 3.8 provides a calculation in terms of the irreducible Brauer characters of $B$.

**Theorem 3.6.** *Let $G$ be a $p$-solvable group and $B \in \mathrm{Bl}(G)$. If $B$ has full defect, then there exists a unique $G$-conjugacy class of characters $\{\theta_1, \dots, \theta_t\} \subseteq \mathrm{Irr}(O_{p'}(G))$ such that $I_G(\theta_1)$ contains a Sylow $p$-subgroup of $G$, and $e_B = \sum_{i=1}^{t} e_{\theta_i}$.*

*Proof.* $B$ covers a unique conjugacy class of $p$-blocks of $O_{p'}(G)$. Since the $p$-blocks of $O_{p'}(G)$ are precisely $\{\{\theta\} \mid \theta \in \mathrm{Irr}(O_{p'}(G))\}$, we have that there exists a unique $G$-conjugacy class $\{\theta_1, \dots, \theta_t\} \subseteq \mathrm{Irr}(O_{p'}(G))$ (where $t = |G : I_G(\theta_1)|$) and positive integers $m_\chi$, $\chi \in B$ such that

$$\chi|_{O_{p'}(G)} = m_\chi \sum_{i=1}^{t} \theta_i$$

for all $\chi \in B$. Note that each $\theta_i$ has the same degree.

Since $B$ has full defect, there exists a character $\chi \in B$ such that $p \nmid \chi(1)$. But

$$\chi(1) = m_\chi \sum_{i=1}^{t} \theta_i(1) = m_\chi t\theta_1(1) = m_\chi |G : I_G(\theta_1)| \theta_1(1).$$

In particular, $p \nmid |G : I_G(\theta_1)|$. Hence, $I_G(\theta_1)$ contains a Sylow $p$-subgroup of $G$.

Now, let $\psi = \sum_{i=1}^{t} \theta_i$, so that $\chi|_{O_{p'}(G)} = m_\chi \psi$ for all $\chi \in B$. Then

$$e_B = \left( \frac{1}{|G|} \sum_{g \in G} \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\chi(g^{-1})g \right)^* .$$

But, by Corollary 3.5, $e_B$ has support contained in $O_{p'}(G)$. Hence,

$$\begin{aligned}
e_B &= \left(\frac{1}{|G|}\sum_{g\in O_{p'}(G)}\sum_{\chi\in \mathrm{Irr}(B)}\chi(1)\chi(g^{-1})g\right)^*\\
&= \left(\frac{1}{|G|}\sum_{g\in O_{p'}(G)}\sum_{\chi\in \mathrm{Irr}(B)}m_\chi\psi(1)m_\chi\psi(g^{-1})g\right)^*\\
&= \left(\frac{1}{|G|}\sum_{g\in O_{p'}(G)}\psi(1)\psi(g^{-1})g\sum_{\chi\in \mathrm{Irr}(B)}m_\chi^2\right)^*\\
&= \left(\frac{\psi(1)M}{|G|}\sum_{g\in O_{p'}(G)}\psi(g^{-1})g\right)^* \in Z(FG)
\end{aligned}$$

Where $M$ is the constant $\sum_{\chi\in \mathrm{Irr}(B)} m_\chi^2$.

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{t} e_{\theta_i} &= \sum_{i=1}^{t}\left(\frac{1}{|O_{p'}(G)|}\sum_{g\in O_{p'}(G)}\theta_i(1)\theta_i(g^{-1})g\right)^*\\
&= \left(\frac{\theta_1(1)}{|O_{p'}(G)|}\sum_{g\in O_{p'}(G)}\sum_{i=1}^{t}\theta_i(g^{-1})g\right)^*\\
&= \left(\frac{\theta_1(1)}{|O_{p'}(G)|}\sum_{g\in O_{p'}(G)}\psi(g^{-1})g\right)^* \in Z(FO_{p'}(G)).
\end{aligned}$$

Since the $e_{\theta_i}e_{\theta_j} = \delta_{ij}e_{\theta_i}$, we have that $\sum_{i=1}^{t} e_{\theta_i}$ is an idempotent of $Z(FO_{p'}(G))$ that differs from $e_B$ only by an element of $F$. Hence, since $e_B \neq 0$,

$$e_B = \sum_{i=1}^{t} e_{\theta_i}.$$

□

**Lemma 3.7.** *Let $G$ be a $p$-solvable group, and suppose that $B \in \mathrm{Bl}(G)$ has full defect. Then*

$$e_B = \left( \frac{1}{|G|_{p'}} \sum_{g \in O_{p'}(G)} \sum_{\varphi \in \mathrm{IBr}(B)} \varphi(1)_{p'} \overline{\varphi(g)} g \right)^* \in Z(FG).$$

*Proof.* By [17, Page 55],

$$e_B = \left( \frac{1}{|G|} \sum_{g \in G} \sum_{\chi \in \mathrm{Irr}(B)} \chi(1) \overline{\chi(g)} g \right)^* \in Z(FG).$$

Then, by applying [17, 3.8], [17, 3.6], and [17, 10.14], we obtain

$$\begin{aligned}
e_B &= \left( \frac{1}{|G|} \sum_{g \in G} \sum_{\chi \in \mathrm{Irr}(B)} \chi(1) \overline{\chi(g)} g \right)^* \\
&= \left( \frac{1}{|G|} \sum_{g \in G^0} \sum_{\chi \in \mathrm{Irr}(B)} \chi(1) \overline{\chi(g)} g \right)^* \\
&= \left( \frac{1}{|G|} \sum_{g \in G^0} \sum_{\varphi \in \mathrm{IBr}(B)} \Phi_\varphi(1) \overline{\varphi(g)} g \right)^* \\
&= \left( \frac{1}{|G|} \sum_{g \in G^0} \sum_{\varphi \in \mathrm{IBr}(B)} |G|_p \varphi(1)_{p'} \overline{\varphi(g)} g \right)^* \\
&= \left( \frac{|G|_p}{|G|} \sum_{g \in G^0} \sum_{\varphi \in \mathrm{IBr}(B)} \varphi(1)_{p'} \overline{\varphi(g)} g \right)^* \\
&= \left( \frac{1}{|G|_{p'}} \sum_{g \in O_{p'}(G)} \sum_{\varphi \in \mathrm{IBr}(B)} \varphi(1)_{p'} \overline{\varphi(g)} g \right)^* .
\end{aligned}$$

□

**Corollary 3.8.** *Let $G$ be a $p$-solvable group. Suppose $B \in \mathrm{Bl}(G)$ has full defect and that $\varphi(1)_{p'} \equiv 1$ modulo $p$ for all $\varphi \in \mathrm{IBr}(B)$. Then*

$$e_B = \left( \frac{1}{|G|_{p'}} \right)^* \sum_{g \in O_{p'}(G)} \sum_{\varphi \in \mathrm{IBr}(B)} \overline{\varphi(g)}^* g \in Z(FG).$$

*In particular, for all $g \in G^0 \setminus O_{p'}(G)$, we have*

$$\sum_{\varphi \in \mathrm{IBr}(B)} \varphi(g)^* = 0 \in F.$$

*Proof.* By Lemma 3.7,

$$\begin{aligned} e_B &= \left( \frac{1}{|G|_{p'}} \sum_{g \in O_{p'}(G)} \sum_{\varphi \in \mathrm{IBr}(B)} \varphi(1)_{p'} \overline{\varphi(g)} g \right)^* \in Z(FG) \\ &= \left( \frac{1}{|G|_{p'}} \right)^* \sum_{g \in O_{p'}(G)} \sum_{\varphi \in \mathrm{IBr}(B)} \overline{\varphi(g)}^* g \in Z(FG). \end{aligned}$$

Hence, if $g \in G^0 \setminus O_{p'}(G)$, the coefficient of $g$ in $e_B$ is 0. But this coefficient is $\left( \frac{1}{|G|_{p'}} \right)^* \sum_{\varphi \in \mathrm{IBr}(B)} \overline{\varphi(g)}^*$. □

### 3.2 Blocks Containing a Linear Brauer Character

We can say more about the block idempotent of a block which contains a linear Brauer character. In fact, it can be realized as the character idempotent (in $FG$) of a linear character of $O_{p'}(G)$, and we can find the corresponding character.

**Lemma 3.9.** *Let $B$ be a block of a $p$-solvable group $G$. If $B$ contains a linear Brauer character, then there exists a unique $\eta_B \in \mathrm{Irr}(O_{p'}(G))$ such that $\varphi|_{O_{p'}(G)} = \varphi(1)\eta_B$ for all $\varphi \in \mathrm{IBr}(B)$. Furthermore, $\eta_B(1) = 1$, $\eta_B$ is invariant under the action by conjugation of $G$ on $\mathrm{Irr}(O_{p'}(G))$, $\mathrm{Irr}(B) = \mathrm{Irr}(G \mid \eta_B)$, and $\mathrm{IBr}(B) = \mathrm{IBr}(G \mid \eta_B)$.*

*Proof.* Suppose $\psi \in \mathrm{IBr}(B)$ is linear. Then $\psi|_{O_{p'}(G)}$ is a linear irreducible character $\lambda \in \mathrm{Irr}(O_{p'}(G))$. Set $\eta_B = \lambda$. It is apparent that $\eta_B(1) = 1$. By [17, Corollary (8.7)], $\eta_B$ is $G$-invariant. Hence, the block $\{\eta_B\} \in \mathrm{Bl}(O_{p'}(G))$ is $G$-invariant. By [17, Theorem (9.2)], if $\varphi \in \mathrm{IBr}(B)$, then every irreducible constituent of $\varphi|_{O_{p'}(G)}$ lies in a $G$-conjugate of the block $\{\eta_B\}$. Hence, $\eta_B$ is the only irreducible constituent of $\varphi|_{O_{p'}(G)}$. This implies that $\varphi$ is a multiple of $\eta_B$. Hence, $\varphi|_{O_{p'}(G)} = \varphi(1)\eta_B$ for all $\varphi \in \mathrm{IBr}(B)$. The fact that $\mathrm{Irr}(B) = \mathrm{Irr}(G \mid \eta_B)$ and $\mathrm{IBr}(B) = \mathrm{IBr}(G \mid \eta_B)$ follows from [17, Theorem (10.20)]. □

**Definition 3.10.** For $B$ a block of a $p$-solvable group $G$ such that $B$ contains a linear Brauer character, define $\eta_B$ to be the unique irreducible character of $O_{p'}(G)$ given by Lemma 3.9 above.

**Theorem 3.11.** *Let $B$ be a block of a $p$-solvable group $G$. If $B$ contains a linear Brauer character, then $e_B = e_{\eta_B}$.*

*Proof.* By [17, Corollary (9.3)], $B$ covers a unique $G$-conjugacy class of blocks of $O_{p'}(G)$. Since the block $\{\eta_B\}$ is $G$-invariant, this means that

$$e_{\eta_B} = \sum_{i=1}^{t} e_{B_i}$$

where $\{B = B_1, \dots, B_t\}$ is the set of blocks of $G$ which cover $\{\eta_B\}$. But, by [17, Theorem (10.20)], $B$ is the unique block of $G$ which covers $\{\eta_B\}$. Hence $e_B = e_{\eta_B}$. □

In particular, Theorem 3.11 implies that if $B$ contains a linear Brauer character, then, since $e_B$ is the idempotent afforded by a *linear* character of $O_{p'}(G)$, the support of $e_B$ in $G$ is exactly $O_{p'}(G)$.

**Corollary 3.12.** *Let $B$ be a block of a $p$-solvable group $G$. If $B$ contains a linear Brauer character, then*

$$\sum_{\varphi \in \mathrm{IBr}(B)} \varphi(g)^* \neq 0$$

*if and only if $g \in O_{p'}(G)$. In particular, if $G$ is a $p$-solvable group and $B_0$ is the principal block of $B$, then*

$$O_{p'}(G) = \left\{ g \in G^0 \ : \ \sum_{\varphi \in \mathrm{IBr}(B_0} \varphi(g)^* \neq 0 \right\}.$$

*Proof.* Let $g \in G^0$. By Theorem 3.8, $\sum_{\varphi \in \mathrm{IBr}(B)} \varphi(g)^* = 0$ if $g \notin O_{p'}(G)$. Now suppose $g \in O_{p'}(G)$. By the proof of Theorem 3.8, the coefficient of $G$ in $e_B$ is $\sum_{\varphi \in \mathrm{IBr}(B)} \varphi(g)^*$. Since $e_B = e_{\eta_B}$ and $\eta_B$ is a linear character, the coefficient of $g$ in $e_{\eta_B}$ is nonzero. Hence, $\sum_{\varphi \in \mathrm{IBr}(B)} \varphi(g)^* \neq 0$ if $g \in O_{p'}(G)$. □

**Theorem 3.13.** *Let $G$ be a $p$-solvable group, and suppose $B \in \mathrm{Bl}(G)$ contains a linear Brauer character. Then the block algebra $FGe_B$ is isomorphic to the principal block algebra $F[G/O_{p'}(G)]$. In particular, $\dim_F(FGe_B) = |G : O_{p'}(G)|$.*

*Proof.* Let $\varphi \in \mathrm{IBr}(B)$ be linear. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi^0 = \varphi$. Such a character $\chi$ exists by the Fong-Swan Theorem [17, Theorem (10.1)]. Then by Lemma 3.9, $\chi|_{O_{p'}(G)} = \varphi|_{O_{p'}(G)} = \eta_B$. Consider the function $\rho_0 : G \to F[G/O_{p'}(G)]$ given by $\rho_0(g) = \chi(g)^* gO_{p'}(G)$ for all $g \in G$. Then $\rho_0$ extends to a surjective vector space homomorphism $\rho_1 : FG \to F[G/O_{p'}(G)]$ in the obvious way. Furthermore, since $\chi$ is linear,

$$\rho_1(g)\rho_1(h) = (\chi(g)^* gO_{p'}(G))(\chi(h)^* hO_{p'}(G)) = \chi(gh)^* ghO_{p'}(G) = \rho_1(gh)$$

for all $g, h \in G$. Hence, $\rho_1$ is a surjective homomorphism of $F$-algebras.

Now, note that by Theorem 3.11,

$$\begin{aligned}
\rho_1(e_B) = \rho_1(e_{\eta_B}) = \rho_1 &\left(\left(\frac{1}{|O_{p'}(G)|}\right)^* \sum_{g \in O_{p'}(G)} \eta_B(g^{-1})^* g\right) \\
&= \left(\frac{1}{|O_{p'}(G)|}\right)^* \sum_{g \in O_{p'}(G)} \eta_B(g^{-1})^* \chi(g)^* gO_{p'}(G) \\
&= \left(\frac{1}{|O_{p'}(G)|}\right)^* \sum_{g \in O_{p'}(G)} \eta_B(g^{-1})^* \eta_B(g)^* gO_{p'}(G) \\
&= \left(\frac{1}{|O_{p'}(G)|}\right)^* |O_{p'}(G)|^* O_{p'}(G) \\
&= O_{p'}(G) \in F[G/O_{p'}(G)].
\end{aligned}$$

Thus, $\rho_1$ maps $e_B$ to the identity of the algebra $F[G/O_{p'}(G)]$. So $\rho_1(xe_B) = \rho_1(x)$ for all $x \in FG$. Thus, if $xe_B = ye_B \in FGe_B$, then $\rho_1(x) = \rho_1(y)$. Therefore, we may define a surjective algebra homomorphism $\rho : FGe_B \to F[G/O_{p'}(G)]$ by $\rho(xe_B) = \rho_1(x)$ for all $x \in FG$.

It remains to show that $\rho$ is injective. We will show the existence of a left inverse to $\rho$. Consider the map $\delta_0 : G/O_{p'}(G) \to FGe_B$ given by $\delta_0(gO_{p'}(G)) = \chi(g^{-1})^* ge_B$. We must show that $\delta_0$ is a well-defined function. First, note that if $z \in O_{p'}(G)$, then

$$\begin{aligned}
\chi(z^{-1})^* ze_B &= \eta_B(z^{-1})^* ze_{\eta_B} \\
&= \eta_B(z^{-1})^* z \left(\frac{1}{|O_{p'}(G)|}\right)^* \sum_{g \in O_{p'}(G)} \eta_B(g^{-1})^* g \\
&= \left(\frac{1}{|O_{p'}(G)|}\right)^* \sum_{g \in O_{p'}(G)} \eta_B(g^{-1})^* \eta_B(z^{-1})^* zg \\
&= \left(\frac{1}{|O_{p'}(G)|}\right)^* \sum_{g \in O_{p'}(G)} \eta_B((zg)^{-1})^* zg \\
&= e_{\eta_B} \\
&= e_B.
\end{aligned}$$

Now, suppose $gO_{p'}(G) = hO_{p'}(G)$. Then $g = hz$ for some $z \in O_{p'}(G)$. Hence,

$$\begin{aligned}
\delta_0(gO_{p'}(G)) = \chi(g^{-1})^* ge_B &= \chi(z^{-1}h^{-1})^* hze_B \\
&= \chi(h^{-1})^* h\chi(z^{-1})^* ze_B \\
&= \chi(h^{-1})^* he_B \\
&= \delta_0(hO_{p'}(G))
\end{aligned}$$

by the previous observation. Hence, $\delta_0$ is well-defined. Then, $\delta_0$ may be extended linearly to a vector space homomorphism $\delta : F[G/O_{p'}(G)] \to FGe_B$. Now, if $g, h \in G$, then

$$\begin{aligned}
\delta(gO_{p'}(G))\delta(hO_{p'}(G)) &= \chi(g^{-1})^* ge_B\chi(h^{-1})^* he_B \\
&= \chi(h^{-1})^* \chi(g^{-1})^* ghe_B^2 \\
&= \chi((gh)^{-1})^* ghe_B \\
&= \delta(ghO_{p'}(G)).
\end{aligned}$$

Hence, $\delta$ is an $F$-algebra homomorphism. Finally, for all $g \in G$,

$$\begin{aligned}\delta(\rho(ge_B)) &= \delta(\rho_0(g)) \\ &= \delta(\chi(g)^* gO_{p'}(G)) \\ &= \chi(g)^* \delta(gO_{p'}(G)) \\ &= \chi(g)^* \chi(g^{-1})^* ge_B \\ &= ge_B.\end{aligned}$$

Since $FGe_B$ is spanned by $\{ge_B \mid g \in G\}$, this implies that $\delta$ is a left inverse to $\rho$, and hence $\rho$ is injective. □

Theorem 3.13 implies that all of blocks of a $p$-solvable group which contain a linear character are isomorphic to the principal block. This implies that any two such blocks containining a linear Brauer character have the same number of Brauer characters and (counting multiplicities) the same Brauer character degrees.

We conclude this section with a brief investigation of blocks containing only linear irreducible Brauer characters. The following lemma is well-known, but the proof is included here for completeness.

**Lemma 3.14.** *If $G$ is a finite group, then* $\bigcap_{\varphi \in \mathrm{IBr}(G)} \ker(\varphi) = O_p(G)$.

*Proof.* Let $I = \bigcap_{\varphi \in \mathrm{IBr}(G)} \ker(\varphi)$. By [17, Lemma (2.32)], $O_p(G) \subseteq I$. Clearly, $I$ is a normal subgroup of $G$. We will show that $I$ is a $p$-subgroup. Let $g \in I$ be a $p$-regular element. Then, by [17, Lemma (6.11)], $\varphi(g) = \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. Hence, for all $\chi \in \mathrm{Irr}(G)$, we have that

$$\chi(g) = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(g) = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(1) = \chi(1).$$

Hence, $g \in \bigcap_{\chi \in \mathrm{Irr}(G)} \ker(\chi) = \{1\}$. Thus, $g = 1$ and $I$ consists of $p$-elements. Hence, $I$ is a normal $p$-subgroup, so $I \subseteq O_p(G)$. □

**Theorem 3.15.** *Let $B$ be a block of a $p$-solvable group $G$ and that every irreducible Brauer character in $B$ is linear. Then the number of irreducible Brauer characters in $B$, is $|G : O_{p'}(G)|_{p'} = |G : O_{p'p}(G)|$.*

*Proof.* By Theorem 3.13, $FGe_B$ is isomorphic to $F[G/O_{p'}(G)]$. Hence, $B$ has the same number of irreducible Brauer characters as $G/O_{p'}(G)$ and every irreducible Brauer character of $G/O_{p'}(G)$ is linear. Hence, to count the number of irreducible Brauer characters in $B$, it suffices to count the number of irreducible Brauer characters of the group $G/O_{p'}(G)$.

Now, since $\mathrm{IBr}(G/O_{p'}(G))$ consists of linear Brauer characters, we have that

$$\big(G/O_{p'}(G)\big)' \subseteq \bigcap_{\varphi \in \mathrm{IBr}(G/O_{p'}(G))} \ker\varphi = O_p(G/O_{p'}(G)) = O_{p'p}(G)/O_{p'}(G)$$

by Lemma 3.14. Hence,

$$\big(G/O_{p'}(G)\big) \,/\, \big(O_{p'p}(G)/O_{p'}(G)\big) \simeq G/O_{p'p}(G)$$

is an abelian $p'$-group with the same number of irreducible Brauer characters as $G/O_{p'}(G)$. Thus,

$$|\,\mathrm{IBr}(G/O_{p'}(G))| = |\,\mathrm{IBr}(G/O_{p'p}(G))| = |G : O_{p'p}(G)| = |G : O_{p'}(G)|_{p'}.$$

$\square$

We say that a block is *basic* if it contains only linear Brauer characters. By Theorems 2.7 and 3.13, such a block is isomorphic to $F[G/O_{p'}(G)]$ and is its own basic algebra.

CHAPTER 4
BASIC ALGEBRAS OF BLOCKS OF $P$-SOLVABLE GROUPS

We are interested in the basic algebra of a $p$-block of a finite $p$-solvable group. Let $p$ be a prime number. Throughout, let $G$ be a finite $p$-solvable group, and use the notation of Chapter 2.

## 4.1 The Basic Algebra of a Block of a $p$-Solvable Group

The basic algebra of a block of a finite $p$-solvable group has a concrete description in terms of primitive idempotents of a $p$-complement and Fong characters.

**Theorem 4.1.** *Let $G$ be a $p$-solvable group and $B \in \mathrm{Bl}(G)$. Let $H \in \mathrm{Hall}_{p'}(G)$. For each $\varphi \in \mathrm{IBr}(B)$, let $\alpha_\varphi \in \mathrm{Irr}(H)$ be a Fong character for $\varphi$, and let*

$$d_\varphi = \left(\frac{1}{|H|}\right)^* \sum_{h\in H} \alpha_\varphi(1)^* \alpha_\varphi\left(h^{-1}\right)^* h \in Z(FH),$$

*the primitive idempotent of $FH$ associated to $\alpha_\varphi$. Let $f_\varphi$ be a primitive idempotent in the algebra $FHd_\varphi$. Then the algebra $xFGx$ is a basic algebra for $FGe_B$, where $x = \sum_{\varphi\in \mathrm{IBr}(B)} f_\varphi$.*

*Proof.* For each $\varphi \in \mathrm{IBr}(B)$, let $M_\varphi$ be the irreducible $FG$-module which affords $\varphi$, and let $V_\varphi$ be a projective cover of $M_\varphi$. Then the basic algebra of $B$ is

$$E = \mathrm{End}_{FG}\left(\bigoplus_{\varphi\in\mathrm{IBr}(B)} V_\varphi\right)^{\mathrm{op}}.$$

(Note that since the $V_i$ are in the block $B$, we have that

$$\mathrm{End}_{FGe_B}\left(\bigoplus_{\varphi\in\mathrm{IBr}(B)} V_\varphi\right)^{\mathrm{op}} = \mathrm{End}_{FG}\left(\bigoplus_{\varphi\in\mathrm{IBr}(B)} V_\varphi\right)^{\mathrm{op}}.)$$

Now, by Lemma 2.26, the projective indecomposable character $\Phi_\varphi$ afforded by $V_\varphi$ is induced from $\alpha_\varphi$. Since the algebra $FH$ is semisimple, $FHd_\varphi$ is a full matrix algebra with identity $d_\varphi$. Since $f_\varphi$ is a primitive idempotent of $FHd_\varphi$, the left ideal $FHf_\varphi$ is an irreducible $FH$-module affording character $\alpha_\varphi$. Hence,

$$V_\varphi \simeq \mathrm{Ind}_{FH}^{FG}(FHf_\varphi) \simeq FG \otimes_{FH} FHf_\varphi \simeq FGf_\varphi.$$

Note that each $f_\varphi$ lives in a different minimal two-sided ideal of $FH$, so the $f_\varphi$ are pairwise orthogonal idempotents.

Therefore,

$$E \simeq \mathrm{End}_{FG}\left(\bigoplus_{\varphi\in \mathrm{IBr}(B)} FGf_\varphi\right)^{\mathrm{op}} = \mathrm{End}_{FG}\left(FGx\right)^{\mathrm{op}} \simeq xFGx$$

by Lemma 2.2. □

**Definition 4.2.** Let $B \in \mathrm{Bl}(G)$, where $G$ is a finite group. $B$ is a *prime power block* if $\varphi(1)$ is a power of $p$ for all $\varphi \in \mathrm{IBr}(B)$.

**Corollary 4.3.** *Let $G$ be a $p$-solvable group and $B \in \mathrm{Bl}(G)$ be a prime power block. Let $H \in \mathrm{Hall}_{p'}(G)$. For each $\varphi \in \mathrm{IBr}(B)$, let $\alpha_\varphi \in \mathrm{Irr}(H)$ be a Fong character for $\varphi$, and let*

$$d_\varphi = \left(\frac{1}{|H|}\right)^* \sum_{h\in H} \alpha_\varphi(1)^* \alpha_\varphi\left(h^{-1}\right)^* h \in Z(FH),$$

*the primitive idempotent of FH associated to $\alpha_\varphi$. Then the algebra xFGx is a basic algebra for $FGe_B$, where $x = \sum_{\varphi\in\mathrm{IBr}(B)} d_\varphi$.*

*Proof.* As $\alpha_\varphi$ has degree $\varphi(1)_{p'} = 1$, the algebra $FHd_\varphi$ is a one-dimensional algebra. Hence, $d_\varphi$ is a primitive idempotent of $FHd_\varphi$. Now apply Theorem 4.1. □

Any idempotent constructed as in Theorem 4.1 will be henceforth referred to as a *basic idempotent* for the block algebra $FGe_B$ or for the block $B$.

Note that if $x$ is a basic idempotent for $FGe_B$, then $xFGx$ is in fact a subalgebra of $FGe_B$.

### 4.2 Blocks of $p$-Nilpotent Groups

In [2], the basic algebra of a nilpotent block with abelian defect group is noted to be isomorphic to the group algebra over the defect group of the block. In this section, we present an elementary (at the level of group characters) proof of a special case of this result.

We will discuss the special case of a block of a $p$-nilpotent group with abelian quotient. Let $G$ be a finite group which has a normal Hall $p'$-subgroup which contains the derived subgroup $G'$. Hence, $G \simeq H \rtimes P$, where $P$ is an abelian $p$-group and $H$ is a $p'$-group. Observe that in this case, the set of $p$-regular elements of $G$ is precisely $H$. Hence, as functions, $\varphi_H = \varphi$ for all $\varphi \in \mathrm{IBr}(G)$.

**Lemma 4.4.** *There is a one-to-one correspondence* $\Phi$ *between the G-conjugacy classes of* $\mathrm{Irr}(H)$ *and* $\mathrm{IBr}(G)$ *given by*

$$\Phi\left(\{\psi_1, \ldots, \psi_t\}\right) = \sum_{i=1}^{t} \psi_i.$$

*Furthermore, each block* $B \in \mathrm{Bl}(G)$ *contains a unique irreducible Brauer character* $\varphi = \sum_{i=1}^{t} \psi_i$*, and the G-conjugates of p-Sylow subgroups of* $I_G(\psi_1)$ *are the defect groups of B.*

*Proof.* Let $\{\psi = \psi_1, \ldots, \psi_t\} \subseteq \mathrm{Irr}(H)$ be a $G$-conjugacy class. Let $I = I_G(\psi)$ be the inertia subgroup of $\psi$. By Fong's Theorem [17, Theorem (10.20)], there is a unique block $b$ of $I$ which covers the block $\{\psi\}$ of $H$ and $\mathrm{IBr}(b) = \mathrm{IBr}(I \mid \psi)$, and the defect groups of $b$ are the Sylow subgroups of $I$. Then, by Green's Theorem [17, Theorem (8.11)], $\mathrm{IBr}(b) = \{\psi\}$, where $\psi$ is now viewed as an irreducible Brauer character of $I$. Now, by the Fong-Reynolds Theorem [17, Theorem (9.14)], there is a unique block, namely $B = b^G$ which covers the block $\{\psi\}$ of $H$. Furthermore, every defect group of $b$ is a defect group of $B$, and $B$ contains a unique irreducible Brauer character, namely $\varphi = \psi^G$. Applying Green's Theorem again, we conclude that $\varphi = \sum_{i=1}^{t} \psi$. Hence, $\Phi$ is well-defined.

Now, if $\varphi \in \mathrm{IBr}(G)$, define $\Gamma(\varphi)$ to be the sum of the irreducible constituents of $\varphi_H$. Then it is clear that $\Gamma$ is an inverse to $\Phi$. $\square$

We can now discuss the basic algebras of the blocks of $G$. Fix $\psi \in \mathrm{Irr}(H)$, and let $I = I_G(\psi)$. Then, by Lemma 4.4, $\psi$ determines a block $B \in \mathrm{Bl}(G)$ with $\mathrm{IBr}(B) = \varphi$ where $\varphi = \sum_{i=1}^{t} \psi_i$ where $\{\psi = \psi_1, \ldots, \psi_t\}$ is the $G$-conjugacy class of $\psi$ in $\mathrm{Irr}(H)$. Now, since

$\varphi(1) = t\psi(1) = |G : I|\psi(1)$ and $G/I$ is a $p$-group and $p \nmid \psi(1)$, we have that $\psi$ is a Fong character for $\varphi$. Let

$$e = \left(\frac{1}{|H|}\right)^{*} \sum_{h \in H} \psi(1)^{*}\psi(h^{-1})^{*} h \in Z(FH),$$

the primitive idempotent of $Z(FH)$ corresponding to $\psi$. Then $FHe$ is a full matrix algebra (of dimension $\psi(1)^2$). By Theorem 4.1, if $x$ is a primitive idempotent in the algebra $FHe$, then $xFGx$ is a basic algebra for the block $B \in \mathrm{Bl}(G)$ which contains $\varphi$. We will show that we may choose a particular primitive idempotent $d$ in $FHe$ in a very special way. First, we need a general linear algebra lemma.

**Lemma 4.5.** *Let $V \neq 0$ be a finite dimensional vector space over $F$. Suppose $T_1, \dots, T_r \in \mathrm{End}_F(V)$ are nilpotent linear transformations which commute pairwise. Then there exists a projection $S \in \mathrm{End}_F(V)$ with $\dim_F(S(V)) = 1$ (in other words, $S$ is a projection onto a one-dimensional subspace of $V$) and $T_i \circ S = 0$ for all $1 \leq i \leq r$.*

*Proof.* Suppose this is false. Among all counterexamples, choose $V$ and $T_1, \dots, T_r \in \mathrm{End}_F(V)$ with $r$ as small as possible. First, will we show that $r > 1$. Indeed, if $r = 0$, then we may choose $S$ to be a projection onto any one-dimensional subspace of $V$ since $V \neq 0$, and we are not in a counterexample. If $r = 1$, then $T_1$ has a nontrivial kernel, and we may choose $S$ to be projection onto any one-dimensional subspace of $\ker(T_1)$, and we are not in a counterexample. Hence, $r > 1$.

By the minimality of $r$, there exists a projection $S_0 \in \mathrm{End}_F(V)$ with $\dim_F(S_0(V)) = 1$ such that $T_i \circ S_0 = 0$ for $1 \leq i \leq r-1$. Thus,

$$0 \neq S_0(V) \subseteq \bigcap_{i=1}^{r-1} \ker(T_i).$$

In particular, $K = \bigcap_{i=1}^{r-1} \ker(T_i) \neq 0$.

Since $T_r$ commutes with $T_1, \dots, T_{r-1}$, we have that

$$T_r(\ker(T_i)) \subseteq \ker(T_i)$$

for all $1 \leq i \leq r-1$. In particular, $T_r(K) \subseteq K$. Since $T_r$ is nilpotent, $\ker(T_r|_K) \neq 0$, since otherwise, every power of $T_r|_K$ is an isomorphism of $K$ and this would yield $K = 0$, which is a contradiction. Therefore, $K \cap \ker(T_r) \neq 0$. Finally,

$$0 \neq \ker(T_r) \cap K = \bigcap_{i=1}^{r} \ker(T_i).$$

Thus, choosing $S_1$ a projection onto a one-dimensional subspace of $\bigcap_{i=1}^{r} \ker(T_i)$ gives $T_i \circ S_1 = 0$ for all $1 \leq i \leq r$, which is a contradiction, since the choice of $V, T_1, \ldots, T_r$ is a counterexample to the statement of the lemma. □

If $g \in I$, then $e^g = e$. Since conjugation by $g$ is a group automorphism of $H$, the map

$$\kappa_g : FHe \to FHe$$

given by

$$\kappa_g(x) = x^g$$

for all $x \in FHe$ is an $F$-algebra automorphism. By the Noether-Skolem Theorem [10, Theorem 6.7], $\kappa_g$ is an inner automorphism of $FHe$. Hence, there exists an element $\hat{g} \in (FHe)^\times$ such that $\kappa_g(x) = \hat{g}^{-1} x \hat{g}$ for all $x \in FHe$.

The next lemma gives an additional condition on $\hat{g}$ so that it is uniquely determined by $g$. Let $P_0 = C_P(H)$ and $P_1 = I_P(\psi)$. Then $P_0 \leq P_1 \leq I$. By Lemma 4.4, $P_1$ is a defect group of $B$. We should show that the basic algebra of $B$ is isomorphic to $FP_1$.

**Lemma 4.6.** *Write $p^k = [P_1 : P_0]$. For all $g \in P_1$, there exists a unique $\widetilde{g} \in (FHe)^\times$ such that $\widetilde{g}^{p^k} = e$ and $x^g = x^{\widetilde{g}}$ for all $x \in FHe$. Furthermore, the map $P_1 \to (FHe)^\times$ given by $g \mapsto \widetilde{g}$ is a group homomorphism. In particular, $\widetilde{g}\widetilde{h} = \widetilde{hg}$ and $\widetilde{g}^{-1} = \widetilde{g^{-1}}$ for all $g, h \in P_1$.*

*Proof.* We will first prove existence. Let $\hat{g} \in (FHe)^\times$ be such that $x^g = x^{\hat{g}}$ for all $x \in FHe$. Then, as $g^{p^k} \in P_0$, we have that $x^{\hat{g}^{p^k}} = x^{g^{p^k}} = x$ for all $x \in FHe$. Thus, since $FHe$ is a full matrix algebra, $\hat{g}^{p^k} \in Z(FHe)$, and so $\hat{g}^{p^k} = \alpha e$ for some $\alpha \in F^\times$. Let $\widetilde{g} = \beta \hat{g}$ where $\beta \in F^\times$ is such that $\beta^{p^k} = \alpha^{-1}$. Then $\widetilde{g}^{p^k} = e$.

Now we prove uniqueness. Suppose $\widetilde{g}_1, \widetilde{g}_2 \in (FHe)^\times$ and $x^{\widetilde{g}_1} = x^{\widetilde{g}_2}$ for all $x \in FHe$. Then $x^{\widetilde{g}_1 \widetilde{g}_2^{-1}} = x$ for all $x \in FHe$. Therefore, $\widetilde{g}_1 \widetilde{g}_2^{-1} \in Z(FHe)$, so that $\widetilde{g}_1 = \gamma \widetilde{g}_2$, some $\gamma \in F^\times$. If $\widetilde{g}_1^{p^k} = e = \widetilde{g}_2^{p^k}$, then

$$e = \widetilde{g}_1^{p^k} = (\gamma \widetilde{g}_2)^{p^k} = \gamma^{p^k} e.$$

Thus, $\gamma$ is a $p^k$-th root of unity in $F$. Since $F$ has characteristic $p$, $\gamma = 1$, and thus $\widetilde{g}_1 = \widetilde{g}_2$.

We will now show that $\widetilde{h}^{-1} = \widetilde{h^{-1}}$ for all $h \in P_1$. It is clear that $x^{\widetilde{h}^{-1}} = x^{h^{-1}} = x^{\widetilde{h^{-1}}}$ for all $x \in FHe$. Now,

$$\left(\widetilde{h}^{-1}\right)^{p^k} = \left(\widetilde{h}^{-1}\right)^{p^k} e = \left(\widetilde{h}^{-1}\right)^{p^k} \left(\widetilde{h}\right)^{p^k} = \left(\widetilde{h}^{-1}\widetilde{h}\right)^{p^k} = e^{p^k} = e.$$

By uniqueness of $\widetilde{h^{-1}}$, this implies that $\widetilde{h}^{-1} = \widetilde{h^{-1}}$.

We will now show that $\widetilde{g}\widetilde{h} = \widetilde{h}\widetilde{g}$ for all $g, h \in P_1$. Indeed, for all $x \in FHe$,

$$x^{\widetilde{g}} = x^g = x^{h^{-1}gh} = x^{\widetilde{h^{-1}}\widetilde{g}\widetilde{h}} = x^{\widetilde{h}^{-1}\widetilde{g}\widetilde{h}}.$$

Furthermore,

$$(\widetilde{h}^{-1}\widetilde{g}\widetilde{h})^{p^k} = \widetilde{h}^{-1}\widetilde{g}^{p^k}\widetilde{h} = \widetilde{h}^{-1} e_K \widetilde{h} = e = \widetilde{g}^{p^k}.$$

Hence, by uniqueness, $\widetilde{h}^{-1}\widetilde{g}\widetilde{h} = \widetilde{g}$.

Finally, we will show that the map $g \mapsto \widetilde{g}$ is a group homomorphism $P_1 \to (FHe)^\times$. Let $g, h \in P_1$. We will show that $\widetilde{g}\widetilde{h} = \widetilde{gh}$. By the first part of the proof (existence and uniqueness), it suffices to show that $x^{\widetilde{g}\widetilde{h}} = x^{gh}$ for all $x \in FHe$ and $(\widetilde{g}\widetilde{h})^{p^k} = e$. If $x \in P_1$, we have

$$x^{\widetilde{g}\widetilde{h}} = (x^{\widetilde{g}})^{\widetilde{h}} = (x^g)^h = x^{gh}.$$

Also,

$$(\widetilde{g}\widetilde{h})^{p^k} = \widetilde{g}^{p^k}\widetilde{h}^{p^k} = ee = e$$

since $\widetilde{g}$ and $\widetilde{h}$ commute. $\square$

Therefore, by Lemma 4.6, for each $g \in P_1$, we may define $\widetilde{g}$ to be the unique element of $(FHe)^\times$ such that $\widetilde{g}^{p^k} = e$ (where $p^k = |P_1 : P_0|$ and $x^g = x^{\widetilde{g}}$ for all $x \in FHe$.

**Lemma 4.7.** *There exists a primitive idempotent $d \in FHe$ such that $\widetilde{g}d = d$ for all $g \in P_1$.*

*Proof.* Let $p^k = [P_1 : P_0]$. Let $C = \{\widetilde{g} - e \mid g \in P_1\} \subset FHe$. By Lemma 4.6, the elements of $C$ commute pairwise. Since $F$ has characteristic $p$ and $e \in Z(FHe)$, we have

$$(\widetilde{g} - e)^{p^k} = \widetilde{g}^{p^k} - e^{p^k} = e - e = 0,$$

so every element of $C$ is nilpotent. Hence, by Lemma 4.5, there exists a primitive idempotent $d \in FHe$ such that $(\widetilde{g} - e)d = 0$ for all $g \in P_1$. Hence,

$$0 = (\widetilde{g} - e)d = \widetilde{g}d - ed = \widetilde{g}d - d$$

for all $g \in P_1$. $\square$

**Lemma 4.8.** *For all $x \in FP_1$, $dxd = xd$ . In particular,*

$$dFP_1d = FP_1d,$$

*and $FP_1d$ is an algebra with identity $d$.*

*Proof.* It suffices to show that $dgd = gd$ for all $g \in P_1$. If $g \in P_1$, then

$$dgd = gg^{-1}dgd = g\widetilde{g}^{-1}d\widetilde{g}d = gdd = gd$$

by Lemma 4.7. $\square$

**Lemma 4.9.** *We have*

$$FP_1d \simeq FP_1$$

*as $F$-algebras.*

*Proof.* Consider the map

$$\Psi : FP_1 \to FP_1d$$

given by $\Psi(x) = xd$ for all $x \in FP_1$. Then it is clear that $\Psi$ is a surjective homomorphism of vector spaces. Hence,

$$\dim_F(FP_1d) \leq \dim_F(FP_1) = |P_1|.$$

Then, the set $\{gd \mid g \in P_1\} \subseteq FP_1d$ is a collection of $|P_1|$ linearly independent elements of $FP_1d$ since $g_1d$ and $g_2d$ involve different elements of $G$ for $g_1 \neq g_2 \in P_1$. Hence,

$$\dim_F(FP_1d) \geq |P_1|$$

and so $\dim_F(FP_1d) = |P_1|$. Therefore, $\Psi$ is an isomorphism of vector spaces.

We will show that $\Psi$ is an algebra homomorphism. Indeed, if $x, y \in FP_1$, then by Lemma 4.8,

$$\Psi(x)\Psi(y) = xdyd = xyd = \Psi(xy).$$

Since, $\Psi(1) = d$, which is the identity of $FP_1d$, $\Psi$ is an algebra homomorphism. $\square$

**Lemma 4.10.** *If $x \in G \setminus I$, then*

$$dxd = 0.$$

*Proof.* First, observe that if $x \in G \setminus I$, then $xex^{-1}$ is a primitive idempotent of $Z(FH)$ which is different from $e$. Hence,

$$exe = exe(x^{-1}x) = e(xex^{-1})x = 0x = 0.$$

So we have

$$dxd = dexed = 0.$$

$\square$

**Lemma 4.11.** *If $y \in H$, then $dyd$ is an $F$-multiple of $d$.*

*Proof.* Since $d$ is an idempotent in $FH$, we have, by Lemma 2.3, that

$$dFHd \simeq (\mathrm{End}_{FH}(FHd))^{\mathrm{op}}.$$

As $FHd$ is a simple $FH$-module and $F$ is algebraically closed, $\mathrm{End}_{FH}(FHf) \simeq F$ by Schur's Lemma [17, Lemma (1.4)]. Thus,

$$dFHd \simeq F^{\mathrm{op}} \simeq F.$$

In particular, $\dim_F(dFHd) = 1$. Since $d \neq 0$, $dFHd = \mathrm{span}_F(d)$, and we are done. $\square$

**Lemma 4.12.** *We have*

$$dFGd = FP_1d.$$

*Proof.* Let $X = \{dxyd \mid x \in H, y \in P\}$. Since $G = H \rtimes P$, $dFGd = \mathrm{span}_F(X)$. We will show that $\mathrm{span}_F(X) = FP_1d$. Let $x \in H$. Let $y \in P \setminus P_1$. Then, since $xy \notin I$, $dxyd = 0$ by Lemma 4.10. If $y \in P_1$, then

$$dxyd = dxdyd = \alpha yd$$

for some $\alpha \in F$ by Lemmas 4.8 and 4.11. Hence, $\mathrm{span}_F(X) = FP_1d$. $\square$

We have proved the final result of this section.

**Theorem 4.13.** *Let $B \in \mathrm{Bl}(G)$, where $G = H \rtimes P$, where $p \nmid |H|$ and $P$ is a $p$-group. Then $B$ contains a unique irreducible Brauer character $\varphi$. Let $\psi$ be an irreducible constituent of $\varphi_H$. Then $B$ has defect group $D = I_P(\psi)$. If $D$ is abelian, then the basic algebra of $B$ is $FD$. Furthermore, the block algebra*

$$FGe_B \simeq \mathrm{Mat}(\psi(1), FD)$$

*as $F$-algebras.*

## 4.3 Basic Idempotents for the Principal Block With Two Irreducible Brauer Characters of a $p$-Solvable Group

Now, we will restrict our attention to the principal block $B$ of $G$ and refine the results of Section 4.1 in the case that $B$ contains exactly two irreducible Brauer characters. We start by stating [18, Theorem 3.1], which classifies $p$-solvable groups whose principal block contains exactly two irreducible Brauer characters.

**Theorem 4.14** (Ninomaya and Wada)**.** *Let $G$ be a $p$-solvable group and $B_0$ the principal block of $FG$. Then the following are equivalent.*

- *The number of irreducible Brauer characters in $B_0$ is 2.*
- *$G = O_{p'pp'p}(G)$ and*
  1. *if $G$ has $p$-length 1, then $p$ is odd and $|G : O_{p'p}(G)| = 2$, and*
  2. *if $G$ has $p$-length 2, then one of the following holds:*
     (a) *$p = 2$, $G/O_{2'2(G)} \simeq E_{3^2} \rtimes \mathbb{Z}_8$.*
     (b) *$p = 2$, $G/O_{2'2(G)} \simeq E_{3^2} \rtimes Q_8$, where $Q_8$ is the quaternion group of order 8.*
     (c) *$p = 2$, $G/O_{2'2(G)} \simeq E_{3^2} \rtimes S_{16}$, where $S_{16}$ is the semidihedral group of order 16.*
     (d) *$p = 2$, $G/O_{2'2(G)} \simeq \mathbb{Z}_q \rtimes \mathbb{Z}_{2^n}$, where $q = 2^n + 1$ is a Fermat prime.*
     (e) *$p = 2^n - 1$ (a Mersenne prime), $G/O_{2'2(G)} \simeq E_{2^n} \rtimes \mathbb{Z}_p$.*

It is also worth stating [11, Theorem A], which implies that in a solvable group, if a block contains exactly two irreducible Brauer characters, then the $p'$-parts of the degrees of these two characters are equal.

**Theorem 4.15** (Isaacs)**.** *Let $G$ be solvable and suppose that $B$ is a $p$-block of $G$. Assume that $\mathrm{IBr}(B) = \{\alpha, \beta\}$ and that $\beta(1) > \alpha(1)$. Then either $p$ is a Mersenne prime and $\beta(1)/\alpha(1) = p$, or else $p = 2$ and $\beta(1)/\alpha(1)$ is a power of 2 that is equal to 8, or else is of the form $q - 1$, where $q$ is a Fermat prime. In all cases, $|G|$ is even.*

We start with a lemma which will allow us to deal with the $p$-length 1 case.

**Lemma 4.16.** *Suppose $G$ is a $p$-solvable group with principal block $B$ and that $G/O_{p'p}(G)$ is an abelian group. Then $B$ is basic.*

*Proof.* Let $\varphi \in \mathrm{IBr}(B)$. By [17, Theorem (10.20)], $O_{p'}(G) \subseteq \ker(\varphi)$. Hence, by [17, Lemma (2.32)], $O_p(G/O_{p'}(G)) = O_{p'p}(G)/O_{p'}(G)$ is contained in the kernel of $\varphi$ viewed as a Brauer character of $G/O_{p'}(G)$. It follows that $O_{p'p}(G) \subseteq \ker(\varphi)$. As $G/O_{p'p}(G)$ is abelian, all of the Brauer characters in $B$ are linear. Hence, by Lemma 2.7, $B$ is basic. $\square$

**Corollary 4.17.** *Suppose $G$ is a $p$-solvable group and that $B$ is the principal block of $G$. If $G$ has $p$-length 1 and $B$ contains exactly two Brauer characters of $G$, then $B \simeq F[G/O_{p'}(G)]$ and $B$ is basic.*

*Proof.* By 3.13, $B \simeq F[G/O_{p'}(G)]$. Then, by 4.14, $G/O_{p'p}(G) \simeq \mathbb{Z}_2$ and it follows from Lemma 4.16 that $B$ is basic. □

The $p$-length 1 case is now complete. We will now discuss the $p$-length 2 case.

**Lemma 4.18.** *Suppose $K \simeq Q \rtimes P$ where $Q$ is an abelian $p'$-group and $P$ is a $p$-group acting transitively on $Q^{\times}$. Then $K$ has $p$-Brauer character table*

| $K^0 = Q$ | $1$ | $Q^{\times}$ |
|---|---|---|
| $1_Q$ | $1$ | $1$ |
| $\varphi$ | $\lvert Q\rvert - 1$ | $-1.$ |

*Furthermore,* $\varphi = \sum_{\chi \in \mathrm{Irr}(Q) \setminus \{1_Q\}} \chi$.

*Proof.* It is clear that $K^0 = Q$ and that $Q^{\times}$ is a $K$-conjugacy class. Also, since $P$ acts transitively on $Q^{\times}$ and $Q$ is an abelian group, we have that the action of $P$ on $\mathrm{Irr}(Q)$ is transitive on the nontrivial characters in $\mathrm{Irr}(Q)$. Let $\varphi \in \mathrm{IBr}(K)$, $\varphi \neq 1_{K^0}$. Let $\chi \in \mathrm{Irr}(Q) \setminus \{1_Q\}$. By Green's Theorem [17, Theorem (8.11)], there exists a *unique* $\varphi \in \mathrm{IBr}(K)$ lying over $\chi$. Furthermore, $\varphi_Q$ is the sum of the distinct $K$-conjugates of $\chi$. As any nontrivial Brauer character lies over $\chi$, $\varphi$ is the unique nontrivial Brauer character of $K$. Furthermore, as $\varphi = \varphi_Q$, we have that $\varphi$ is the sum of the nontrivial irreducible characters of $Q$. Let $1 \neq x \in Q$. We will compute $\varphi(x)$. By the Second Orthogonality Relation [12, Theorem (2.18)],

$$\varphi(x) = \sum_{\chi \in \mathrm{Irr}(Q) \setminus \{1_Q\}} \chi(x) = \sum_{\chi \in \mathrm{Irr}(Q)} \chi(x) - 1_Q(1) = 0 - 1 = -1.$$

As $Q$ is abelian and there are $|Q| - 1$ irreducible constituents of $\varphi_Q$, we have that $\varphi$ has degree $|Q| - 1$. □

**Theorem 4.19.** *Suppose $G$ is a finite $p$-solvable group with $O_{p'}(G) = 1$ and $G/O_p(G) \simeq Q \rtimes P$ where $Q$ is an abelian $p'$-group of order at least 3 and $P$ is a $p$-group acting transitively on $Q^\times$. Then*

- *if $H \in \mathrm{Hall}_{p'}(G)$, then $H \simeq Q$.*
- *for all $H \in \mathrm{Hall}_{p'}(G)$ and all $1_H \neq \chi \in \mathrm{Irr}(H)$, the algebra $xFGx$ is a basic algebra for $FG$, where $x = \sum_{h \in H}(1 + \chi(h)^*)h \in FG$.*

*Proof.* Let $\pi : G \to Q \rtimes P$ be the natural projection. Let $H \in \mathrm{Hall}_{p'}(G)$. It is clear that $\pi|_H : H \to Q$ is an isomorphism.

As $O_p(G) \subseteq \ker(\varphi)$ [17, Lemma (2.32)] for all $\varphi \in \mathrm{IBr}(G)$, the irreducible Brauer characters of $G$ are precisely $\{\alpha \circ \pi \mid \alpha \in \mathrm{IBr}(G/O_p(G))\}$. By Lemma 4.18, $G$ has exactly two irreducible Brauer characters, the trivial character $1_{G^0}$, and a nontrivial Brauer character $\alpha$. Since $O_{p'}(G)$ is trivial, $G$ has a unique block by [17, Theorem (10.20)]. Hence, by Theorem 4.3 (and the fact that $|Q| - 1$ must be a power of $p$), $xFGx$ is a basic algebra for $FG$, where $x \in FG$ is the sum of the Fong idempotents for $1_{G^0}$ and $\alpha$. Clearly, $1_H$ is a Fong character for $1_{G^0}$. By Lemma 4.18, any nontrivial irreducible character of $H$ is a Fong character for $\varphi$. Fix a nontrivial $\chi \in \mathrm{Irr}(H)$. Then

$$\begin{aligned} x &= e_{1_H} + e_\chi \\ &= \left(\frac{1}{|H|}\right)^* \sum_{h \in H} 1_H(1)^* 1_H(h)^* h + \left(\frac{1}{|H|}\right)^* \sum_{h \in H} \chi(1)^* \chi(h)^* h \\ &= \left(\frac{1}{|H|}\right)^* \sum_{h \in H} (1 + \chi(h)^*)h. \end{aligned}$$

Now, since $|Q| \geq 3$ and $P$ acts transitively on $Q^\times$, we have that $1 \neq |Q^\times|$ is a power of $p$. Therefore, $|H| = |Q|$ is congruent to 1 modulo $p$. It follows that $|H|^* = 1$, and we obtain the desired result by the previous line. □

**Corollary 4.20.** *Suppose $G$ is a finite $p$-solvable group with $G/O_{p'p}(G) \simeq Q \rtimes P$ where $Q$ is an abelian $p'$-group of order at least 3 and $P$ is a $p$-group acting transitively on $Q^\times$. Let $B$ be the principal block of $G$. Then*

- *if* $H \in \mathrm{Hall}_{p'}(G/O_{p'}(G))$*, then* $H \simeq Q$*.*
- *for all* $H \in \mathrm{Hall}_{p'}(G/O_{p'}(G))$ *and all* $1_H \neq \chi \in \mathrm{Irr}(H)$*, the algebra* $xF[G/O_{p'}(G)]x$ *is a basic algebra for* $B$*, where*
  $x = \sum_{h \in H}(1 + \chi(h)^*)h \in F[G/O_{p'}(G)]$.

*Proof.* By [20], $B$ is isomorphic to $F[G/O_{p'}(G)]$. Hence, we will show that the group $G/O_{p'}(G)$ satisfies the hypotheses of Theorem 4.19. As $O_{p'}(G/O_{p'}(G))$ is trivial, we have that $O_{p'p}(G/O_{p'}(G)) = O_p(G/O_{p'}(G))$. Hence,

$$\begin{aligned}
&(G/O_{p'}(G))/O_p(G/O_{p'}(G)) \\
&= (G/O_{p'}(G))/(O_{p'p}(G)/O_{p'}(G)) \\
&\simeq G/O_{p'p}(G) \\
&\simeq Q \rtimes P.
\end{aligned}$$

The result is obtained by applying Theorem 4.19 to $G/O_{p'}(G)$. □

**Corollary 4.21.** *Suppose* $G$ *is a* $p$*-solvable group and* $B$ *is the principal block of* $G$*. If* $G$ *has* $p$*-length* 2 *and* $B$ *contains exactly two irreducible Brauer characters, then*

- *if* $H \in \mathrm{Hall}_{p'}(G/O_{p'}(G))$*, then* $H$ *is an elementary abelian group.*
- *for all* $H \in \mathrm{Hall}_{p'}(G/O_{p'}(G))$ *and all* $1_H \neq \chi \in \mathrm{Irr}(H)$*, the algebra* $xF[G/O_{p'}(G)]x$ *is a basic algebra for* $B$*, where*
  $x = \left(\frac{1}{|H|}\right)^* \sum_{h \in H}(1 + \chi(h)^*)h \in F[G/O_{p'}(G)]$.

*Proof.* By [18, Theorem 3.1], $G/O_{p'p}(G) \simeq Q \rtimes P$ where $Q$ is an elementary abelian $p'$-group of order at least 3 and $P$ is a $p$-group. Since $O_{p'p}(G) \subseteq \ker(\varphi)$ for all $\varphi \in B$ (by [17, Lemma (10.20)] and [17, Lemma (2.32)]), the irreducible Brauer characters in $B$ may be viewed as irreducible Brauer characters of $G/O_{p'p}(G)$. By Green's Theorem [17, Theorem (8.11)], $P$ must act transitively on the nontrivial irreducible characters of $Q$, and hence $P$ acts transitively on $Q^{\times}$. The desired result is obtained by applying Corollary 4.20. □

Finally, we summarize our calculation of the basic algebra in the event that the principal block contains exactly two irreducible Brauer characters.

**Theorem 4.22.** *Suppose $G$ is a $p$-solvable group, $B$ is the principal block of $G$, and that $B$ contains exactly two irreducible Brauer characters. Then*

- $B \simeq F[G/O_{p'}(G)]$.
- *If $G$ has $p$-length 1, then $B$ is basic.*
- *If $G$ has $p$-length 2, then $B$ has basic algebra $xF[G/O_{p'}(G)]x$, where $x = \sum_{h \in H}(1 + \chi(h)^*)h \in F[G/O_{p'}(G)]$ for any $H \in \mathrm{Hall}_{p'}(G/O_{p'}(G))$ and any $1_H \neq \chi \in \mathrm{Irr}(H)$. Necessarily, $H$ is elementary abelian.*

**Corollary 4.23.** *Suppose $G$ is a $p$-solvable group and $B$ is a block of $G$ containing exactly two irreducible Brauer characters, at least one of which is linear. Then*
*$B$ has basic algebra $F[G/O_{p'}(G)]$ if both Brauer characters in $B$ are linear.*
*If $B$ contains a non-linear Brauer character, $B$ has basic algebra $xF[G/O_{p'}(G)]x$, where $x = \sum_{h \in H}(1 + \chi(h)^*)h \in F[G/O_{p'}(G)]$ for any $H \in \mathrm{Hall}_{p'}(G/O_{p'}(G))$ and any $1_H \neq \chi \in \mathrm{Irr}(H)$.*

*Proof.* Follows immediately from Theorems 4.22 and 3.13. □

# CHAPTER 5
# IMPLEMENTATION IN GAP

In this chapter, we discuss the implementation of Theorem 4.1 in GAP (see [8]).

## 5.1 The Algorithm

In theory, the algorithm for computing a basic idempotent $x \in FG$ of a block $B$ of a finite $p$-solvable group $G$ is just the proof of Theorem 4.1. We outline the general process here and then discuss the technical details of the implementation.

**Algorithm 5.1.** *INPUT:* A prime $p$, a finite $p$-solvable group $G$, an algebraically closed field $F$ of characteristic $p$, and a $p$-block $B \in \mathrm{Bl}(G)$.

*OUTPUT:* An idempotent $x$ in $FG$ such that $xFGx$ is a basic algebra of $B$.

1. Choose a Hall $p'$-subgroup $H$ of $G$.
2. For each $\varphi \in \mathrm{IBr}(B)$, consider $\varphi_H$. Choose an irreducible constituent $\alpha_\varphi$ of $\varphi_H$ with $\alpha_\varphi(1) = \varphi(1)_{p'}$.
3. Compute the primitive idempotent
$$d_\varphi = \left(\frac{1}{|H|}\right)^* \sum_{h \in H} \alpha_\varphi(h^{-1})^* h$$
of $Z(FH)$ corresponding to $\alpha_\varphi$.
4. Find a primitive idempotent $f_\varphi$ in the full matrix algebra $FHd_\varphi$
5. Set $x = \sum_{\varphi \in \mathrm{IBr}(B)} d_\varphi$.

One difficulty of implementing the above algorithm in GAP is the nature of the field $F$. As one can not implement algebraic closures of fields in GAP, a finite field must suffice. We say that a finite field $F$ is *large enough for $G$* if $F$ contains all of the $|G|_{p'} = |H|$-roots of unity. As the values of the irreducible Brauer characters of $G$ are sums of $|H|$-th roots of unity, such a field will suffice for our calculations. Hence, we take $F$ to be the field `GF(p^n)` where $n$ is the smallest integer such that $|H|$ divides $p^n - 1$.

In the GAP implementation of the above algorithm, we must also compute $d_\varphi$ as in Step 3. The algebraic integer $|H|^{-1}$ has a natural embedding ($|H|^{-1}$ mod $p$) in the prime field of $F$), but $\alpha_\varphi(h^{-1})^*$ does not necessarily have such an embedding. However, this is not a major problem. Notice that $\alpha_\varphi|_{\langle h^{-1}\rangle}$ is a positive integer linear combination of linear characters $\sum_{i=1}^{s} a_i\lambda_i$, $a_i > 0$, $\lambda_i \in \mathrm{Irr}(\langle h^{-1}\rangle)$ is a linear character. Since $\lambda_i$ is a linear character, we have that $\lambda_i(h^{-1})$ is a $|H|$-th root of unity, and is hence equal to some power of $\mathtt{E}(|\mathtt{H}|)$, where $E(m) = e^{\frac{2\pi i}{m}}$. Extending the correspondence $E(|H|) \mapsto Z(p^n)^{\frac{p^n-1}{|H|}}$ (where $\mathtt{Z}(\mathtt{p}^\mathtt{n})$ is GAP's preferred generator for $F^\times$) to powers of $E(|H|)$ gives an embedding of $\lambda_i(h^{-1})$ in $F$ in the following way: if $\lambda_i(h^{-1}) = \mathtt{E}(|\mathtt{H}|)^\mathtt{w} \in \mathbb{C}$, set $\lambda_i(h^{-1})^* = \left(\mathtt{Z}(\mathtt{p}^\mathtt{n})^{\frac{\mathtt{p}^\mathtt{n}-1}{|\mathtt{H}|}}\right)^\mathtt{w} \in F$. Thus,

$$\alpha_\varphi\left(h^{-1}\right)^* = \sum_{i=1}^{s} a_i^*\lambda_i\left(h^{-1}\right)^* = \sum_{i=1}^{s}(a_i \bmod p)\lambda_i\left(h^{-1}\right)^*.$$

Once we have computed $d_\varphi$, the problem remains of finding a primitive idempotent $f_\varphi \in FHd_\varphi$. In [4], the authors discuss a method for computing primitive idempotents in matrix algebras, but this method is currently not implemented. For small examples, the following brute force algorithm is effective for finding a primitive idempotent $f_\varphi$. Note that by Wedderburn's Theorem, $FHd_\varphi \simeq \mathrm{End}_F(M_{\alpha_\varphi})$, where $M$ is the irreducible $FH$-module whose character is $\alpha_\varphi$. Equivalently, $M_{\alpha_\varphi}$ is an irreducible $FH$-submodule (or irreducible left ideal) of $FHd_\varphi$.

**Algorithm 5.2.** *INPUT:* The full matrix algebra $FHd_\varphi$, the Fong character $\alpha_\varphi$.
*OUTPUT:* A primitive idempotent $f_\varphi \in FHd_\varphi$.

1. Let $x$ be a random element of $FHd_\varphi$.
2. If $\dim_F(FHd_\varphi x) = \alpha_\varphi(1)^2 - \alpha_\varphi(1)$, then $M = FHd_\varphi/FHd_\varphi x$ is an irreducible $FHd_\varphi$-module. Otherwise, go back to Step 1.
3. Let $X \subseteq M$ be a basis, and let $Z \subseteq FHd_\varphi$ be a generating set. Let $\Lambda$ be the algebra isomorphism $\Lambda : FHd_\varphi \to \mathrm{End}_F(M)$ given by $\Lambda(z)$ is the matrix with respect to the basis $X$ for the action of $z$ on $M$ given by left multiplication.

4. Let $P$ be the $\alpha_\varphi(1) \times \alpha_\varphi(1)$ matrix over $F$ with 1 in the upper left-hand corner and 0 everywhere else. Then $P$ is a primitive idempotent of $\mathrm{End}_F(I)$.
5. Set $f_\Phi = \Lambda^{-1}(P)$.

Another issue with implementing Algorithm 5.1 in GAP is finding a $p$-complement $H$. In GAP version 4.5.6, functionality for computing a $p$-complement exists via the `HallSubgroup` or `SylowComplement` commands. However, in the current state of GAP, these commands are only functional for groups which are solvable. Hence, in the GAP implementation, we can only calculate the basic algebra for groups which are solvable, and not the larger class of $p$-solvable groups.

## 5.2 Source Code

This section contains the source code for the GAP implementation of Algorithm 5.1. The code may be input into GAP using the `Read` function. The main function is `BasicAlgebra(G,B,p)` which takes as input a finite solvable group $G$, a prime $p$, and a $p$-block $B$ given as an element in the list `BlocksInfo(CharacterTable(G,p))`. The function `BasicAlgebra(G,B,p)` outputs a GAPrecord containing the following components:

- `group` is the group $G$,
- `pcomp` is a $p$-complement subgroup $H$,
- `prime` is the prime $p$,
- `block` is the $p$-block $B$,
- `field` is a field $F$ of characteristic $p$ which is large enough for $G$,
- `groupalgebra` is the group algebra $FG$,
- `basidem` is an idempotent $x \in FG$ such that $xFGx$ is a basic algebra of $B$, and
- `basicalgebra` is the basic algebra $xFGx$ of $B$.

Here is the source code.

```
#GAP SOURCE CODE FOR BASIC ALGEBRA PROGRAM


ModPReduction:=function(x,sizeF,sizeH)
#computes the mod-p reduction in F of x, which is a |H|-th root of 1
    local p,n,zeta,beta;
    p:=FactorsInt(sizeF)[1];
    n:=Log(sizeF,p);
    zeta:=Z(p^n)^((p^n-1)/sizeH);
    beta:=0;
    while beta >= 0 do
        if x = E(sizeH)^beta then
            break;
        fi;
        if x <> E(sizeH)^beta then
            beta:=beta+1;
        fi;
    od;
    return zeta^beta;
end;


pPrimePart:=function(n,p)
#computes the p'-part of the integer n
    local factors,ppart,x;
    factors:=FactorsInt(n);
    if not p in factors then
        return n;
    fi;
```

```
    ppart:=1;
    for x in factors do
        if x = p then
            ppart:=ppart*x;
        fi;
    od;
    return n/ppart;
end;

FongIdemMod:=function(FG,G,H,phi,p)
#computes a Fong idempotent for a Brauer character phi of G
    local alpha,constituents,theta,f_alpha,FH,FHf,basisM,pi,M,X,J,count,
     dimensiondeccheck,rightdimensioncheck,FHf_alpha,x,j,fongdeg,sizeF,k,a,s,
     matak,lambda,imagebasisI,coeffsakxj,h,f,F,basisI,primidemmat,Phi,MatAlg,
     proj,imgproj,gensFHf,z,imgPhi,i;
    fongdeg:=pPrimePart(DegreeOfCharacter(phi),p);
    constituents:=ConstituentsOfCharacter(RestrictedClassFunction(phi,H));
    for theta in constituents do
        if DegreeOfCharacter(theta)=fongdeg then
            alpha:=Irr(H)[Position(Irr(H),theta)];
            break;
        fi;
    od;
    sizeF:=RootInt(Size(FG),Dimension(FG));
    f_alpha:=(One(FG)/Size(H)*One(FG))*Sum(Elements(H),h->DegreeOfCharacter(alpha)
     *One(FG) * Sum(ConstituentsOfCharacter(RestrictedClassFunction(
     alpha,Subgroup(H,[h]))), lambda->ScalarProduct(CharacterTable(Subgroup(H,[h])),
```

```
 lambda,RestrictedClassFunction(alpha,Subgroup(H,[h])))*One(FG)*ModPReduction(
 h^lambda,sizeF,Order(H))) * (h^-1)^Embedding(G,FG));
FH:=Subalgebra(FG,Image(Embedding(G,FG),Elements(H)));
FHf:=FH*f_alpha;
if Dimension(FHf) = 1 then
    return f_alpha;
fi;
F:=UnderlyingField(FG);
count:=0;
rightdimensioncheck:=0;
dimensiondeccheck:=0;
for x in FHf do #find a maximal submodule of regular module
    J:=LeftIdeal(FHf,[x]);
    count:=count+1;
    if Dimension(J) = (fongdeg^2-fongdeg) then
        dimensiondeccheck:=dimensiondeccheck+1;
        break;
    fi;
od;
rightdimensioncheck:=rightdimensioncheck+1;
MatAlg:=FullMatrixAlgebra(F,fongdeg);
pi:=NaturalHomomorphismBySubspace(FHf,J);
M:=Image(pi);
gensFHf:=GeneratorsOfAlgebra(FHf);
basisM:=Basis(M);
imgPhi:=[];
for k in [1..Length(gensFHf)] do
```

```
        matak:=IdentityMat(fongdeg,F);
        for j in [1..Length(basisM)] do
            coeffsakxj:=Coefficients(basisM,(gensFHf[k]*PreImagesRepresentative(
             pi,basisM[j]))^pi);
            for i in [1..Length(basisM)] do
                matak[i][j]:=coeffsakxj[i];
            od;
        od;
        imgPhi[k]:=matak;
    od;
    Phi:=AlgebraHomomorphismByImages(FHf,MatAlg,gensFHf,imgPhi);
    proj:=IdentityMat(fongdeg,F);
    for s in [2..fongdeg] do
        proj[s][s]:=Zero(F);
    od;
    return PreImage(Phi,proj);
end;

BasIdem:=function(FG,G,H,B,p)
#computes a basic idempotent of B as a sum of Fong idempotents
    local e,i;
    e:=Sum(B.modchars,i->FongIdemMod(FG,G,H,Irr(G,p)[i],p));
    return e;
end;

PComplement:=function(G,p)
#computes a Hall p'-subgroup of G
```

```
    local pi, hallsub,K,phi;
    K:=SmallGroup(IdSmallGroup(G));
    pi:=Set(FactorsInt(Order(G)));
    if not p in pi then
        return G;
    fi;
    if Size(pi)=1 then
        return TrivialSubgroup(G);
    fi;
    Remove(pi,Position(pi,p));
    phi:=IsomorphismGroups(K,G);
    hallsub:=HallSubgroup(K,pi);
    hallsub:=Image(phi,hallsub);
    return hallsub;
end;

BasicAlgebra:=function(G,B,p)
#main function which computes the basic algebra of B
    local F,FG,H,n,bas,basidem;
    H:=PComplement(G,p);
    n:=1;
    while n >= 1 do
        if (p^n-1)/Size(H) in Integers then
            break;
        fi;
        if not (p^n-1)/Size(H) in Integers then
            n:=n+1;
```

```
        fi;
    od;
    F:=GF(p^n);
    FG:=GroupRing(F,G);
    basidem:=BasIdem(FG,G,H,B,p);
    bas:=rec(group:=G,block:=B,prime:=p,field:=F,groupalgebra:=FG,
     idempotent:=basidem,basicalgebra:=basidem*FG*basidem,pcomp:=H);
    return bas;
end;
```

### 5.3 Sample Output

In this section, we use the GAP code given in Section 5.2 to produce a list of blocks $B$ of *solvable* groups $G$ satisfying the following.

- $|G| < 96$,
- $G$ is not $p$-nilpotent or $B$ has nonabelian defect groups,
- $B$ is not basic, and
- if $B$ contains a linear Brauer character, then $B$ is the principal block of $G$.

Table 5-1 gives the GAP Small Group ID (given by `IdSmallGroup(G)`), the prime $p$, the block ID of $B$ (which is the position of $B$ in `BlocksInfo(CharacterTable(G,p))`), and the dimension of $B$ and its basic algebra $bas(B)$.

Table 5-1. Dimensions of Blocks and Their Basic Algebras in Small Solvable Groups

| Small Group ID | $p$ | Block ID | Brauer degrees | $\dim_F(B)$ | $\dim_F(bas(B))$ |
|---|---|---|---|---|---|
| $[24, 4]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[24, 6]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[24, 8]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[24, 12]$ | 2 | 1 | $[1, 2]$ | 24 | 11 |
| $[24, 12]$ | 3 | 2 | $[3]$ | 9 | 1 |
| $[24, 12]$ | 3 | 3 | $[3]$ | 9 | 1 |
| $[30, 3]$ | 3 | 2 | $[2]$ | 12 | 3 |
| $[30, 3]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[30, 3]$ | 5 | 2 | $[2]$ | 20 | 5 |
| $[40, 4]$ | 5 | 3 | $[2]$ | 20 | 5 |
| $[40, 6]$ | 5 | 3 | $[2]$ | 20 | 5 |
| $[40, 8]$ | 5 | 3 | $[2]$ | 20 | 5 |
| $[42, 5]$ | 3 | 2 | $[2]$ | 12 | 3 |
| $[42, 5]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[42, 5]$ | 3 | 4 | $[2]$ | 12 | 3 |
| $[42, 5]$ | 7 | 2 | $[2]$ | 28 | 7 |
| $[48, 5]$ | 3 | 5 | $[2]$ | 12 | 3 |
| $[48, 5]$ | 3 | 6 | $[2]$ | 12 | 3 |
| $[48, 6]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[48, 6]$ | 3 | 4 | $[2]$ | 12 | 3 |
| $[48, 6]$ | 3 | 5 | $[2]$ | 12 | 3 |
| $[48, 7]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[48, 7]$ | 3 | 4 | $[2]$ | 12 | 3 |
| $[48, 7]$ | 3 | 5 | $[2]$ | 12 | 3 |
| $[48, 8]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[48, 8]$ | 3 | 4 | $[2]$ | 12 | 3 |
| $[48, 8]$ | 3 | 5 | $[2]$ | 12 | 3 |
| $[48, 10]$ | 3 | 5 | $[2]$ | 12 | 3 |
| $[48, 10]$ | 3 | 6 | $[2]$ | 12 | 3 |
| $[48, 12]$ | 3 | 5 | $[2]$ | 12 | 3 |
| $[48, 12]$ | 3 | 6 | $[2]$ | 12 | 3 |
| $[48, 13]$ | 3 | 5 | $[2]$ | 12 | 3 |
| $[48, 13]$ | 3 | 6 | $[2]$ | 12 | 3 |
| $[48, 14]$ | 3 | 5 | $[2]$ | 12 | 3 |
| $[48, 14]$ | 3 | 6 | $[2]$ | 12 | 3 |
| $[48, 15]$ | 2 | 2 | $[2]$ | 32 | 8 |
| $[48, 15]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[48, 15]$ | 3 | 4 | $[2, 2]$ | 24 | 6 |
| $[48, 16]$ | 2 | 2 | $[2]$ | 32 | 8 |
| $[48, 16]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[48, 16]$ | 3 | 4 | $[2, 2]$ | 24 | 6 |
| $[48, 17]$ | 2 | 2 | $[2]$ | 32 | 8 |
| $[48, 17]$ | 3 | 3 | $[2]$ | 12 | 3 |
| $[48, 17]$ | 3 | 4 | $[2, 2]$ | 24 | 6 |

Table 5-1. Continued

| Small Group ID | $p$ | Block ID | Brauer degrees | $\dim_F(B)$ | $\dim_F(bas(B))$ |
|---|---|---|---|---|---|
| [48, 18] | 2 | 2 | [2] | 32 | 8 |
| [48, 18] | 3 | 3 | [2] | 12 | 3 |
| [48, 18] | 3 | 4 | [2, 2] | 24 | 6 |
| [48, 19] | 3 | 5 | [2] | 12 | 3 |
| [48, 19] | 3 | 6 | [2] | 12 | 3 |
| [48, 28] | 2 | 1 | [1, 2] | 48 | 22 |
| [48, 28] | 3 | 2 | [2, 2] | 24 | 6 |
| [48, 28] | 3 | 3 | [3] | 9 | 1 |
| [48, 28] | 3 | 4 | [3] | 9 | 1 |
| [48, 29] | 2 | 1 | [1, 2] | 48 | 22 |
| [48, 29] | 3 | 2 | [2, 2] | 24 | 6 |
| [48, 29] | 3 | 3 | [3] | 9 | 1 |
| [48, 29] | 3 | 4 | [3] | 9 | 1 |
| [48, 30] | 2 | 1 | [1, 2] | 48 | 22 |
| [48, 30] | 3 | 3 | [3] | 9 | 1 |
| [48, 30] | 3 | 4 | [3] | 9 | 1 |
| [48, 30] | 3 | 5 | [3] | 9 | 1 |
| [48, 30] | 3 | 6 | [3] | 9 | 1 |
| [48, 34] | 3 | 5 | [2] | 12 | 3 |
| [48, 34] | 3 | 6 | [2] | 12 | 3 |
| [48, 36] | 3 | 5 | [2] | 12 | 3 |
| [48, 36] | 3 | 6 | [2] | 12 | 3 |
| [48, 37] | 3 | 5 | [2] | 12 | 3 |
| [48, 37] | 3 | 6 | [2] | 12 | 3 |
| [48, 38] | 2 | 2 | [2] | 32 | 8 |
| [48, 38] | 3 | 5 | [2, 2] | 24 | 6 |
| [48, 39] | 2 | 2 | [2] | 32 | 8 |
| [48, 39] | 3 | 5 | [2, 2] | 24 | 6 |
| [48, 40] | 2 | 2 | [2] | 32 | 8 |
| [48, 40] | 3 | 5 | [2, 2] | 24 | 6 |
| [48, 41] | 2 | 2 | [2] | 32 | 8 |
| [48, 41] | 3 | 5 | [2, 2] | 24 | 6 |
| [48, 43] | 3 | 5 | [2] | 12 | 3 |
| [48, 43] | 3 | 6 | [2] | 12 | 3 |
| [48, 48] | 2 | 1 | [1, 2] | 48 | 22 |
| [48, 48] | 3 | 3 | [3] | 9 | 1 |
| [48, 48] | 3 | 4 | [3] | 9 | 1 |
| [48, 48] | 3 | 5 | [3] | 9 | 1 |
| [48, 48] | 3 | 6 | [3] | 9 | 1 |
| [56, 3] | 7 | 3 | [2] | 28 | 7 |
| [56, 5] | 7 | 3 | [2] | 28 | 7 |
| [56, 7] | 7 | 3 | [2] | 28 | 7 |
| [60, 3] | 3 | 3 | [2] | 12 | 3 |
| [60, 3] | 3 | 4 | [2] | 12 | 3 |

Table 5-1. Continued

| Small Group ID | $p$ | Block ID | Brauer degrees | $\dim_F(B)$ | $\dim_F(bas(B))$ |
|---|---|---|---|---|---|
| [60, 3] | 3 | 5 | [2] | 12 | 3 |
| [60, 3] | 3 | 6 | [2] | 12 | 3 |
| [60, 3] | 5 | 3 | [2] | 20 | 5 |
| [60, 3] | 5 | 4 | [2] | 20 | 5 |
| [60, 7] | 3 | 3 | [4] | 48 | 3 |
| [60, 7] | 5 | 2 | [2, 2] | 40 | 10 |
| [60, 8] | 3 | 3 | [2, 2] | 24 | 6 |
| [60, 8] | 3 | 4 | [2, 2] | 24 | 6 |
| [60, 8] | 5 | 3 | [2, 2] | 40 | 10 |
| [60, 12] | 3 | 3 | [2] | 12 | 3 |
| [60, 12] | 3 | 4 | [2] | 12 | 3 |
| [60, 12] | 3 | 5 | [2] | 12 | 3 |
| [60, 12] | 3 | 6 | [2] | 12 | 3 |
| [60, 12] | 5 | 3 | [2] | 20 | 5 |
| [60, 12] | 5 | 4 | [2] | 20 | 5 |
| [66, 3] | 3 | 2 | [2] | 12 | 3 |
| [66, 3] | 3 | 3 | [2] | 12 | 3 |
| [66, 3] | 3 | 4 | [2] | 12 | 3 |
| [66, 3] | 3 | 5 | [2] | 12 | 3 |
| [66, 3] | 3 | 6 | [2] | 12 | 3 |
| [66, 3] | 11 | 2 | [2] | 44 | 11 |
| [70, 3] | 5 | 2 | [2] | 20 | 5 |
| [70, 3] | 5 | 3 | [2] | 20 | 5 |
| [70, 3] | 5 | 4 | [2] | 20 | 5 |
| [70, 3] | 7 | 2 | [2] | 28 | 7 |
| [70, 3] | 7 | 3 | [2] | 28 | 7 |
| [72, 4] | 3 | 3 | [2] | 36 | 9 |
| [72, 6] | 3 | 3 | [2] | 36 | 9 |
| [72, 8] | 3 | 3 | [2] | 36 | 9 |
| [72, 15] | 2 | 1 | [1, 2] | 24 | 11 |
| [72, 15] | 2 | 2 | [2, 2, 2] | 48 | 12 |
| [72, 15] | 3 | 2 | [3, 3] | 54 | 6 |
| [72, 22] | 3 | 2 | [2] | 36 | 9 |
| [72, 23] | 3 | 2 | [2] | 36 | 9 |
| [72, 24] | 3 | 2 | [2] | 36 | 9 |
| [72, 26] | 3 | 3 | [2] | 36 | 9 |
| [72, 28] | 3 | 3 | [2] | 36 | 9 |
| [72, 30] | 3 | 3 | [2] | 36 | 9 |
| [72, 31] | 3 | 3 | [2] | 36 | 9 |
| [72, 33] | 3 | 3 | [2] | 36 | 9 |
| [72, 35] | 3 | 3 | [2] | 36 | 9 |
| [72, 40] | 3 | 1 | [1, 1, 1, 1, 2] | 72 | 41 |
| [72, 41] | 3 | 1 | [1, 1, 1, 1, 2] | 72 | 41 |
| [72, 42] | 2 | 1 | [1, 2] | 24 | 11 |

Table 5-1. Continued

| Small Group ID | $p$ | Block ID | Brauer degrees | $\dim_F(B)$ | $\dim_F(bas(B))$ |
|---|---|---|---|---|---|
| [72, 42] | 3 | 2 | [3] | 27 | 3 |
| [72, 42] | 3 | 3 | [3] | 27 | 3 |
| [72, 43] | 2 | 1 | [1, 2] | 24 | 11 |
| [72, 43] | 2 | 2 | [2, 2, 2] | 48 | 12 |
| [72, 43] | 3 | 2 | [3, 3] | 54 | 6 |
| [72, 44] | 2 | 2 | [2, 2, 2] | 48 | 12 |
| [72, 44] | 3 | 2 | [3, 3] | 54 | 6 |
| [78, 5] | 3 | 2 | [2] | 12 | 3 |
| [78, 5] | 3 | 3 | [2] | 12 | 3 |
| [78, 5] | 3 | 4 | [2] | 12 | 3 |
| [78, 5] | 3 | 5 | [2] | 12 | 3 |
| [78, 5] | 3 | 6 | [2] | 12 | 3 |
| [78, 5] | 3 | 7 | [2] | 12 | 3 |
| [78, 5] | 13 | 2 | [2] | 52 | 13 |
| [80, 5] | 5 | 5 | [2] | 20 | 5 |
| [80, 5] | 5 | 6 | [2] | 20 | 5 |
| [80, 6] | 5 | 3 | [2] | 20 | 5 |
| [80, 6] | 5 | 4 | [2] | 20 | 5 |
| [80, 6] | 5 | 5 | [2] | 20 | 5 |
| [80, 7] | 5 | 3 | [2] | 20 | 5 |
| [80, 7] | 5 | 4 | [2] | 20 | 5 |
| [80, 7] | 5 | 5 | [2] | 20 | 5 |
| [80, 8] | 5 | 3 | [2] | 20 | 5 |
| [80, 8] | 5 | 4 | [2] | 20 | 5 |
| [80, 8] | 5 | 5 | [2] | 20 | 5 |
| [80, 10] | 5 | 5 | [2] | 20 | 5 |
| [80, 10] | 5 | 6 | [2] | 20 | 5 |
| [80, 12] | 5 | 5 | [2] | 20 | 5 |
| [80, 12] | 5 | 6 | [2] | 20 | 5 |
| [80, 13] | 5 | 5 | [2] | 20 | 5 |
| [80, 13] | 5 | 6 | [2] | 20 | 5 |
| [80, 14] | 5 | 5 | [2] | 20 | 5 |
| [80, 14] | 5 | 6 | [2] | 20 | 5 |
| [80, 15] | 2 | 2 | [2] | 32 | 8 |
| [80, 15] | 2 | 3 | [2] | 32 | 8 |
| [80, 15] | 5 | 3 | [2] | 20 | 5 |
| [80, 15] | 5 | 4 | [2, 2] | 40 | 10 |
| [80, 16] | 2 | 2 | [2] | 32 | 8 |
| [80, 16] | 2 | 3 | [2] | 32 | 8 |
| [80, 16] | 5 | 3 | [2] | 20 | 5 |
| [80, 16] | 5 | 4 | [2, 2] | 40 | 10 |
| [80, 17] | 2 | 2 | [2] | 32 | 8 |
| [80, 17] | 2 | 3 | [2] | 32 | 8 |
| [80, 17] | 5 | 3 | [2] | 20 | 5 |

Table 5-1. Continued

| Small Group ID | $p$ | Block ID | Brauer degrees | $\dim_F(B)$ | $\dim_F(bas(B))$ |
|---|---|---|---|---|---|
| [80, 17] | 5 | 4 | [2, 2] | 40 | 10 |
| [80, 18] | 2 | 2 | [2] | 32 | 8 |
| [80, 18] | 2 | 3 | [2] | 32 | 8 |
| [80, 18] | 5 | 3 | [2] | 20 | 5 |
| [80, 18] | 5 | 4 | [2, 2] | 40 | 10 |
| [80, 19] | 5 | 5 | [2] | 20 | 5 |
| [80, 19] | 5 | 6 | [2] | 20 | 5 |
| [80, 29] | 5 | 3 | [2, 2] | 40 | 10 |
| [80, 31] | 5 | 3 | [2, 2] | 40 | 10 |
| [80, 33] | 5 | 3 | [2, 2] | 40 | 10 |
| [80, 34] | 5 | 3 | [2, 2] | 40 | 10 |
| [80, 35] | 5 | 5 | [2] | 20 | 5 |
| [80, 35] | 5 | 6 | [2] | 20 | 5 |
| [80, 37] | 5 | 5 | [2] | 20 | 5 |
| [80, 37] | 5 | 6 | [2] | 20 | 5 |
| [80, 38] | 5 | 5 | [2] | 20 | 5 |
| [80, 38] | 5 | 6 | [2] | 20 | 5 |
| [80, 39] | 2 | 2 | [2] | 32 | 8 |
| [80, 39] | 2 | 3 | [2] | 32 | 8 |
| [80, 39] | 5 | 5 | [2, 2] | 40 | 10 |
| [80, 40] | 2 | 2 | [2] | 32 | 8 |
| [80, 40] | 2 | 3 | [2] | 32 | 8 |
| [80, 40] | 5 | 5 | [2, 2] | 40 | 10 |
| [80, 41] | 2 | 2 | [2] | 32 | 8 |
| [80, 41] | 2 | 3 | [2] | 32 | 8 |
| [80, 41] | 5 | 5 | [2, 2] | 40 | 10 |
| [80, 42] | 2 | 2 | [2] | 32 | 8 |
| [80, 42] | 2 | 3 | [2] | 32 | 8 |
| [80, 42] | 5 | 5 | [2, 2] | 40 | 10 |
| [80, 44] | 5 | 5 | [2] | 20 | 5 |
| [80, 44] | 5 | 6 | [2] | 20 | 5 |
| [84, 5] | 3 | 3 | [2] | 12 | 3 |
| [84, 5] | 3 | 4 | [2] | 12 | 3 |
| [84, 5] | 3 | 5 | [2] | 12 | 3 |
| [84, 5] | 3 | 6 | [2] | 12 | 3 |
| [84, 5] | 3 | 7 | [2] | 12 | 3 |
| [84, 5] | 3 | 8 | [2] | 12 | 3 |
| [84, 5] | 7 | 3 | [2] | 28 | 7 |
| [84, 5] | 7 | 4 | [2] | 28 | 7 |
| [84, 8] | 3 | 3 | [2, 2] | 24 | 6 |
| [84, 8] | 3 | 4 | [2, 2] | 24 | 6 |
| [84, 8] | 3 | 5 | [2, 2] | 24 | 6 |
| [84, 8] | 7 | 3 | [2, 2] | 56 | 14 |
| [84, 11] | 2 | 2 | [3] | 36 | 4 |

Table 5-1. Continued

| Small Group ID | $p$ | Block ID | Brauer degrees | $\dim_F(B)$ | $\dim_F(bas(B))$ |
|---|---|---|---|---|---|
| [84, 11] | 2 | 3 | [3] | 36 | 4 |
| [84, 11] | 7 | 2 | [3] | 63 | 7 |
| [84, 14] | 3 | 3 | [2] | 12 | 3 |
| [84, 14] | 3 | 4 | [2] | 12 | 3 |
| [84, 14] | 3 | 5 | [2] | 12 | 3 |
| [84, 14] | 3 | 6 | [2] | 12 | 3 |
| [84, 14] | 3 | 7 | [2] | 12 | 3 |
| [84, 14] | 3 | 8 | [2] | 12 | 3 |
| [84, 14] | 7 | 3 | [2] | 28 | 7 |
| [84, 14] | 7 | 4 | [2] | 28 | 7 |
| [88, 3] | 11 | 3 | [2] | 44 | 11 |
| [88, 5] | 11 | 3 | [2] | 44 | 11 |
| [88, 7] | 11 | 3 | [2] | 44 | 11 |
| [90, 3] | 3 | 2 | [2] | 36 | 9 |
| [90, 3] | 3 | 3 | [2] | 36 | 9 |
| [90, 3] | 5 | 2 | [2] | 20 | 5 |
| [90, 3] | 5 | 3 | [2] | 20 | 5 |
| [90, 3] | 5 | 4 | [2] | 20 | 5 |
| [90, 3] | 5 | 5 | [2] | 20 | 5 |
| [90, 7] | 3 | 2 | [2] | 36 | 9 |
| [90, 7] | 3 | 3 | [2] | 36 | 9 |
| [90, 7] | 5 | 4 | [2] | 20 | 5 |
| [90, 7] | 5 | 5 | [2] | 20 | 5 |
| [90, 7] | 5 | 6 | [2] | 20 | 5 |
| [90, 9] | 3 | 2 | [2] | 36 | 9 |
| [90, 9] | 3 | 3 | [2] | 36 | 9 |
| [90, 9] | 5 | 2 | [2] | 20 | 5 |
| [90, 9] | 5 | 3 | [2] | 20 | 5 |
| [90, 9] | 5 | 4 | [2] | 20 | 5 |
| [90, 9] | 5 | 5 | [2] | 20 | 5 |

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# BIOGRAPHICAL SKETCH

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