MA 437 Hints for problem 6.16 Fall 2014

6.16 If a group G is isomorphic to H, prove that $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(H)$.

Hints: We are given that G is isomorphic to H, so there exists an isomorphism from G onto H. Hence, there is a function $\phi : G \to H$ which is one-to-one, onto, and operation-preserving. We need to show that $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(H)$. Hence, we must exhibit an isomorphism $\Gamma : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ which is one-to-one, onto, and operation-preserving. (Note that Γ is the capital Greek letter pronounced "Gamma.") To find Γ, we must find a way to take any automorphism $\alpha : G \to G$ of G, and construct an automorphism $\Gamma(\alpha) : H \to H$ of H.

To construct Γ, let $\alpha \in \mathrm{Aut}(G)$ and consider the function $\phi\alpha\phi^{-1}$ (recall that the juxtaposition of functions means to compose them from right to left). Since $\phi^{-1} : H \to G$, $\alpha : G \to G$, and $\phi : G \to H$, the function $\phi\alpha\phi^{-1}$ is a function from H to H. We claim that $\phi\alpha\phi^{-1}$ is an *automorphism* of H (that is, you need to prove that $\phi\alpha\phi^{-1}$ is one-to-one, onto, and operation-preserving). Then, we may now define a mapping $\Gamma : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ by $\Gamma(\alpha) = \phi\alpha\phi^{-1}$. There are several steps which remain to show that Γ is an isomorphism:

- Γ **is one-to-one.** Suppose $\alpha_1, \alpha_2 \in \mathrm{Aut}(G)$ and $\Gamma(\alpha_1) = \Gamma(\alpha_2)$. Prove that $\alpha_1 = \alpha_2$. This step will use the fact that, since ϕ is one-to-one and onto, $\phi\phi^{-1}$ is the identity function on H, and $\phi^{-1}\phi$ is the identity function on G.

- Γ **is onto.** Let $\beta \in \mathrm{Aut}(H)$. To show that Γ is onto, we must exhibit an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\Gamma(\alpha) = \beta$. Once you have found the desired function $\alpha : G \to G$, don't forget to prove that α is an automorphism of G.

- Γ **is operation-preserving.** Let $\alpha_1, \alpha_2 \in \mathrm{Aut}(G)$. We must prove that $\Gamma(\alpha_1\alpha_2) = \Gamma(\alpha_1)\Gamma(\alpha_2)$, where we recall that the group operation in $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$ is function composition. To prove this, simply use the definition of Γ:

$$\Gamma(\alpha_1)\Gamma(\alpha_2) = (\phi\alpha_1\phi^{-1})(\phi\alpha_2\phi^{-1}) = \cdots = \Gamma(\alpha_1\alpha_2)$$