1.3.10. We first show that $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n \mid a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace of F^n. We need to show that $\vec{0} \in W_1$, and that W_1 is closed under addition and scalar multiplication. Since $\vec{0} = (0, 0, \ldots, 0) \in F^n$ and $0+0+\cdots+0 = 0$, it follows that $\vec{0} \in W_1$. Now, let $x, y \in W_1$, $c \in F$. Then $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$, where $a_1+a_2+\cdots+a_n = 0$ and $b_1 + b_2 + \cdots + b_n = 0$. Now, $x + y = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$, and

$$\begin{aligned}(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Thus, $x + y \in W_1$. Now, $cx = (ca_1, ca_2, \ldots, ca_n)$, and

$$\begin{aligned}ca_1 + ca_2 + \cdots + ca_n &= c(a_1 + a_2 + \cdots + a_n) \\ &= c0 \\ &= 0.\end{aligned}$$

Thus, $cx \in W_1$. Therefore, W_1 is a subspace of F^n.

Now, we show that $W_2 = \{(a_1, a_2, \ldots, a_n) \in F^n \mid a_1+a_2+\cdots+a_n = 1\}$ is not a subspace of F^n. Since $\vec{0} = (0, 0, \ldots, 0) \in F^n$ and $0 + 0 + \cdots + 0 = 0 \neq 1$, $\vec{0} \notin W_2$. Hence, W_2 is not a subspace of F^n.

1.3.13. Let $V = \{f \in \mathcal{F}(S, F) \mid f(s_0) = 0\}$. Recall that the function $z : S \to F$ defined by $z(s) = 0$ for all $s \in S$ is the zero vector in $\mathcal{F}(S, F)$. Since $z(s_0) = 0$, $z \in V$. Now, let $f, g \in V$. Then $(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0$, and hence $f + g \in V$. Let $f \in V$, $c \in F$. Then $(cf)(s_0) = c(f(s_0)) = c0 = 0$, and hence $cf \in V$. Since the zero vector in $\mathcal{F}(S, F)$ is in V, and V is closed under addition and scalar multiplication, V is a subspace of $\mathcal{F}(S, F)$.

1.4.7. Let $(a_1, a_2, \ldots, a_n) \in F^n$. Then,

$$\begin{aligned}(a_1, a_2, \ldots, a_n) &= (a_1, 0, \ldots, 0) + (0, a_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, a_n) \\ &= a_1(1, 0, \ldots, 0) + a_2(0, 1, 0, \ldots, 0) + \cdots + a_n(0, \ldots, 0, 1) \\ &= a_1e_1 + a_2e_2 + \cdots + a_ne_n \in \mathrm{span}\{e_1, e_2, \ldots, e_n\}.\end{aligned}$$

Thus, any vector in F^n can be written as a linear combination of $e_1, e_2, \ldots, e_n$. Hence, $\{e_1, e_2, \ldots, e_n\}$ generates F^n.

1.4.12. ($\Rightarrow$) Suppose W is a subspace of V. We wish to show that $W = \mathrm{span}(W)$. Let $w \in W$. Then, since $w = 1 \cdot w$, w is a linear combination of vectors in W, and hence $w \in \mathrm{span}(W)$. So $W \subseteq \mathrm{span}(W)$. Now, let $v \in \mathrm{span}(W)$. Then

$$v = a_1w_1 + a_2w_2 + \cdots + a_nw_n$$

for some scalars $a_1, a_2, \ldots, a_n \in F$ and vectors $w_1, w_2, \ldots, w_n \in W$. Since each $w_i \in W$ and W is closed under scalar multiplication, each $a_iw_i \in W$. Now, since W is closed under addition, $a_1w_1 + a_2w_2 + \cdots a_nw_n \in W$, and hence $v \in W$. It follows that $\mathrm{span}(W) \subseteq W$, and thus $W = \mathrm{span}(W)$.

($\Leftarrow$) Now suppose W is a subset of V, and that $W = \mathrm{span}(W)$. We wish to show that W is a subspace of V. By Theorem 1.5, $\mathrm{span}(W)$ is a subspace of V. Since $W = \mathrm{span}(W)$, W is a subspace of V.
