MA 431 – Solutions for Homework 1 Spring 2015

1.2.9. *Proof of Corollary 1.* Suppose there are two zero vectors, say $\vec{0}_1$, $\vec{0}_2 \in V$ such that $x + \vec{0}_1 = x$ and $x + \vec{0}_2 = x$ for all $x \in V$. Since $x + \vec{0}_1 = x = x + \vec{0}_2$ for all $x \in V$, Theorem 1.1 implies that $\vec{0}_1 = \vec{0}_2$, and hence the vector $\vec{0}$ described in (VS 3) is unique.

Proof of Corollary 2. Let $x \in V$. Suppose there are two vectors, $y_1, y_2 \in V$ such that $x + y_1 = \vec{0}$ and $x + y_2 = \vec{0}$. Since $x + y_1 = \vec{0} = x + y_2$, Theorem 1.1 implies that $y_1 = y_2$, and hence the vector y described in (VS 4) is unique.

Proof of Theorem 1.2(c). Let $a \in F$. Then, since $\vec{0} = \vec{0} + \vec{0}$ and $\vec{0} + a\vec{0} = a\vec{0}$ by (VS 3), we have

$$\begin{aligned} \vec{0} + a\vec{0} &= a\vec{0} \\ &= a(\vec{0} + \vec{0}) \\ &= a\vec{0} + a\vec{0} \text{ by (VS 7) }. \end{aligned}$$

Thus, $\vec{0} + a\vec{0} = a\vec{0} + a\vec{0}$. By Theorem 1.1, $\vec{0} = a\vec{0}$.

1.2.12. Let $V = \{f : \mathbb{R} \to \mathbb{R} \mid f(-t) = f(t) \text{ for all } t \in \mathbb{R}\}$. To show that V is a vector space over $\mathbb{R}$, we must show that V is closed under addition and scalar multiplication, and that V satisfies the vector space properties (VS 1)–(VS 8).

Let $f, g \in V$. Then for all $t \in \mathbb{R}$,

$$(f + g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f + g)(-t),$$

and hence $f + g \in V$. Thus, V is closed under addition.

Now, let $c \in \mathbb{R}$, $f \in V$. Then for all $t \in \mathbb{R}$,

$$(cf)(-t) = c(f(-t)) = c(f(t)) = (cf)(t),$$

and hence $cf \in V$. Thus, V is closed under scalar multiplication. We now show that V satisfies (VS 1)–(VS 8)

(VS 1) Let $f, g \in V$. Then, since $(f + g)(t) = f(t) + g(t) = g(t) + f(t) = (g + f)(t)$ for all $t \in \mathbb{R}$, it follows that $f + g = g + f$.

(VS 2) Let $f, g, h \in V$. Then, since

$$[(f + g) + h](t) = (f(t) + g(t)) + h(t) = f(t) + (g(t) + h(t)) = [f + (g + h)](t)$$

for all $t \in \mathbb{R}$, it follows that $(f + g) + h = f + (g + h)$.

(VS 3) Define a function $z : \mathbb{R} \to \mathbb{R}$ by $z(t) = 0$ for all $t \in \mathbb{R}$. Then, since $z(-t) = 0 = z(t)$ for all $t \in \mathbb{R}$, $t \in V$. Now, Let $f \in V$. Since $(f + z)(t) = f(t) + z(t) = f(t)$ for all $t \in \mathbb{R}$, it follows that $f + z = f$.

(VS 4) Let $f \in V$. Define a function $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = -f(t)$ for all $t \in \mathbb{R}$. Then

$$(f + g)(t) = f(t) - f(t) = 0 = z(t)$$

for all $t \in \mathbb{R}$.

(VS 5) Let $f \in V$, then $1f(t) = f(t)$ for all $t \in \mathbb{R}$, so $1f = f$.

(VS 6) Let $a, b \in \mathbb{R}$, $f \in V$. Then

$$(ab)f(t) = a(bf(t))$$

for all $t \in \mathbb{R}$, hence $(ab)f = a(bf)$.

(VS 7) Let $a \in \mathbb{R}$, $f, g \in V$. Then

$$[a(f + g)](t) = a[f(t) + g(t)] = af(t) + bg(t) = (af + bg)(t)$$

for all $t \in \mathbb{R}$, so $a(f + g) = af + ag$.

(VS 8) Let $a, b \in \mathbb{R}$, $f \in V$. Then

$$[(a + b)f](t) = (a + b)f(t) = af(t) + bf(t) = (af + bf)(t)$$

for all $t \in \mathbb{R}$, so $(a + b)f = af + bf$.

1.2.13 No, V is not a vector space over $\mathbb{R}$. Indeed, consider $1, 2 \in \mathbb{R}$, $(3, 4) \in V$. Then

$$(1 + 2)(3, 4) = 3(3, 4) = (3 \cdot 3, 4) = (9, 4).$$

On the other hand,

$$1(3, 4) + 2(3, 4) = (1 \cdot 3, 4) + (2 \cdot 3, 4) = (3, 4) + (6, 4) = (9, 4 \cdot 4) = (9, 16).$$

Thus, $(1 + 2)(3, 4) \neq 1(3, 4) + 2(3, 4)$, so (VS 8) fails.
Note: There are probably a few other properties which fail as well.

1.2.15 No, V is not a vector space over $\mathbb{C}$. Indeed, consider $i \in \mathbb{C}$, $(1, 1, \ldots, 1) \in V$. Note that

$$i(1, 1, \ldots, 1) = (i, i, \ldots, i) \notin V.$$

Thus, V is not closed under scalar multiplication, so V is not a vector space over $\mathbb{C}$.