MA 437 Snow Day Notes for March 6 – Section 2.1 Spring 2015

Recall.

Theorem 2.4. Let V and W be vector spaces over the same field F, and suppose $T : V \to W$ is linear. Then T is one-to-one if and only if $\mathrm{N}(T) = \{\vec{0}_V\}$.

Theorem 2.5. Let V and W be finite-dimensional vector spaces with $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Then the following are equivalent:
(a) T is one-to-one.
(b) T is onto.
(c) $\mathrm{rank}(T) = \dim(V)$.

Note. Theorem 2.5 does not hold for infinite-dimensional vector spaces. The theorem also fails for T not linear.

Example 12. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1).$$

Claim 1. T is linear.

Proof of Claim 1. Let $x, y \in \mathbb{R}^2$, $c \in \mathbb{R}$. Then $x = (a_1, a_2), y = (b_1, b_2)$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Hence,

$$T(cx + y) = T(ca_1 + b_1, ca_2 + b_2) = (ca_1 + b_1 + ca_2 + b_2, ca_1 + b_1).$$

On the other hand,

$$cT(x) + T(y) = cT(a_1, a_2) + T(b_1, b_2) = c(a_1 + a_2, a_1) + (b_1 + b_2, b_1) = (ca_1 + ca_2 + b_1 + b_2, ca_1 + b_1).$$

Thus, $T(cx + y) = cT(x) + T(y)$, so T is linear. □

Claim 2. T is one-to-one.

Proof of Claim 2. Let $(a_1, a_2) \in \mathrm{N}(T)$. Then $T(a_1, a_2) = (0, 0)$, and hence $(a_1 + a_2, a_1) = (0, 0)$, which implies $a_1 + a_2 = 0$ and $a_1 = 0$. Solving this system yields $a_1 = a_2 = 0$, and thus $(a_1, a_2) = (0, 0)$. By Theorem 2.4, T is one-to-one. □

Since T is linear and one-to-one, and since the domain and codomain of T have the same dimension, Theorem 2.5 implies that T is onto.

The next theorem and its corollary illustrate the following remarkable fact about linear transformations: If you know the values of a linear transformation $T : V \to W$ on a basis of V, then you know the value $T(v)$ for any vector $v \in V$!

Theorem 2.6 Let V and W be vector spaces over F, and suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. Then for any vectors $w_1, w_2, \ldots, w_n \in W$, there exists *exactly one* linear transformation $T : V \to W$ such that
$T(v_1) = w_1, T(v_2) = w_2, \ldots, T(v_n) = w_n$.

Main Idea of Proof. Let $x \in V$. Then, since $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, there exist *unique* scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$x = a_1v_1 + a_2v_2 + \cdots a_nv_n.$$

Thus, we may define a function $T : V \to W$ by

$$T(x) = a_1w_1 + a_2w_2 + \cdots a_nw_n,$$

where the $a_1, a_2, \ldots, a_n \in F$ are uniquely determined by x as above.

Exercise. Show that T is linear.

To show the uniqueness of T, suppose that $S : V \to W$ is linear and has $S(v_1) = w_1, S(v_2) = w_2, \ldots, S(v_n) = w_n$.

Then for any $x \in V$,

$$\begin{aligned} S(x) &= S(a_1v_1 + a_2v_2 + \cdots + a_nv_n) \\ &= a_1S(v_1) + a_2S(v_2) + \cdots + a_nS(v_n) \\ &= a_1w_1 + a_2w_2 + \cdots + a_nw_n \\ &= T(x). \end{aligned}$$

Thus, $S = T$ and the uniqueness of T follows. □

The proof of Theorem 2.6 illustrates that if two linear transformations from V to W have the same values on a basis for V, then those transformations must be identical. We state this as a corollary.

Corollary. Let V and W be vector spaces over F. If $\{v_1, v_2, \ldots, v_n\}$ is a basis for V and $S, T : V \to W$ are two linear transformations such that $T(v_1) = S(v_1), T(v_2) = S(v_2), \ldots, T(v_n) = S(v_n)$, then $S = T$. □

Example 14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (2a_2 - a_1, 3a_1)$.

Exercise. Show that T is linear and $\{(1, 2), (1, 1)\}$ is a basis for $\mathbb{R}^2$.

Note that $T(1, 2) = (3, 3)$ and that $T(1, 1) = (1, 3)$.

Suppose that $S : \mathbb{R}^2 \to \mathbb{R}^2$ is *any* linear transformation with $S(1, 2) = (3, 3)$ and $S(1, 1) = (1, 3)$, then by the Corollary, S must be the same function as T!
