Surviving a National Football League Survivor Pool

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Abstract. In this paper, an analytical approach to National Football League (NFL) survival pools is investigated. This paper introduces into the literature NFL survival pools and presents optimization models for determining strategies. Computational results indicate that planning only partway through the season yields the highest survival probabilities, which dominate millions of randomly generated strategies.

1. Introduction

Betting on sports games has grown rapidly in the United States and abroad. According to the American Gaming Association, sports fans were expected to wager over $95 billion on football games in the National Football League (NFL) or at the collegiate level in 2015 alone (American Gaming Association 2015). Some estimate the size of this market to be even greater—for example, the Nevada Gaming Commission in 2011 estimated that nearly $400 billion is wagered on football annually (Green 2012).

Betting pools encompass a portion of this market and come in a variety of flavors. Two popular pools include the National Collegiate Athletic Association (NCAA) basketball tournament and single-week NFL pools. The ever-popular NCAA basketball tournament occurs in early spring, when 68 college basketball teams play a single-elimination tournament. In single-week NFL pools, participants are required to select the winner of each game played that week (13–16 games). The participant who selects the most teams correctly wins the pool, with ties paying out equal pots to all winners.

This paper addresses another popular version of sports betting pools, NFL survival pools, which differ substantially in format from those addressed previously. In an NFL survivor pool, each participant pays an entrance fee and must select a winning team in each week of the NFL season. Each team in the league can only be chosen one time throughout the season. The participant selects the winning team for that week at the beginning of each week’s slate of games. The term “survivor pool” comes from the fact that a participant is eliminated as soon as one week’s selection loses or ties. The goal of the survivor pool is to survive as long as possible into the 17-week season, which requires a participant to be correct in his selection of a winning team each week. This leads to a complex optimization problem where, given probability estimates for the remaining games of the season, a participant must select a team to win while also considering future matchups. The complexity is expanded when participants pay for (a limited number of) multiple entries.

Bettor pools have received attention from the academic community because of their relation with other competitive real-world optimization problems, in addition to the immediate application to sports betting. As described by Clair and Letscher (2007), a sports betting pool is an example of a more general competitive setting where agents must select from a set of choices. Examples given in their paper include airline routes and combinatorial auctions (Clair and Letscher 2007).

More related to the present study is the edge-weighted online bipartite matching problem (EWBMP) (Khuller et al. 1994, Raghvendra 2016, Bansal et al. 2014, Anthony and Chung 2014), a variant of the sequential stochastic assignment problem (SSAP) (Derman et al. 1972; Gershkov and Moldovanu 2010; Nikolaev and Jacobson 2010; Khatibi et al. 2015, 2014). The EWBMP is defined on a bipartite graph \( G = (A \cup B, E) \). The vertices in \( B \) appear sequentially, along with the weights of the edges incident to the vertex and its neighbors in \( A \). Upon the arrival of a vertex \( b \in B \), a vertex \( a \in A \) must be irrevocably assigned to \( b \) with the goal of maximizing (or minimizing) the total weight of all edges selected. Associating the teams with \( A \) and the weeks with \( B \), the decision maker in an NFL survival pool is faced with a similar decision—selecting team \( a \) for
week $b$ has a probability of success, and the goal is to succeed in the selection for every week without knowing ahead of time the probabilities in future weeks. The applications for these problem classes include selection of passengers for airport screening, kidney assignment, and accepting bids in real estate markets (Lee 2009). The literature on the EWBMP and the SSAP has focused on developing algorithms with theoretical performance guarantees assuming a distribution on the stochastic inputs. This paper focuses on data-driven computational models assuming only point estimates on the win probabilities in the future, which move according to external, unpredictable events. Additionally, we study a generalization of the EWBMP, where multiple edges are to be selected by separate agents each time a vertex in $B$ arrives.

Clair and Letscher (2007) address both the NCAA tournament and single-week NFL pools. They present probabilistic models, establish a closed-form solution for expected return in the single-week football pool, and show an approximation to the expected return for a fixed set of picks in the context of the NCAA tournament. The results indicate that when the total number of entries is small, a single participant should select teams most favored to win, but when the number of entries grows, the entrant should pick teams less likely to win in order to gain a competitive advantage over the masses. The present paper incorporates a similar type of strategy, where the amount of “forward thinking” required in making picks and saving teams for later in the season depends on the number of entries. The popularity of the NCAA tournament has resulted in a stream of publications (Kaplan and Garstka 2001, Breiter and Carlin 1997, Kaplan and Magazine 2003, Metrick 1996, Caudill and Godwin 2002, Kvam and Sokol 2006, Stekler and Klein 2012, Niemi et al. 2008). However, the literature on football pools has been limited, despite the growing popularity of single-week pools and NFL survivor pools (RunYourPool.com 2015, OfficePoolStop.com 2015, Politi 2015).

This paper addresses the following question: Given a set of probability estimates for the remaining games of the NFL season, what strategy should a participant employ to maximize the probability of winning a survivor pool? The standard greedy approach amounts to selecting each week the team that has the highest probability of winning among those teams that have not been previously selected. This strategy is often employed because the ultimate goal is to never lose in any given week. The problem, as will be shown through computational experimentation, is that planning only week to week results in obtaining strategies with a low probability of lasting the entire season, in comparison with long-term planning strategies.

For the single-entry case, an integer programming (IP) model is formulated for obtaining such long-term strategies. The solutions obtained are compared with standard greedy approaches and randomly generated strategies. It is shown experimentally that planning only partway through the season leads to strategies that greatly increase the probability of surviving the entire season.

Because of the complex dependencies between the selection decisions made in the multiple-entry case, a more complex model is required to solve the optimization problem. In this case, based on the principle of inclusion-exclusion, a nonlinear optimization model is formulated. This novel model is able to capture the intricate dependencies in order to accurately model the probability of at least one entry surviving the season, or some fixed time horizon. The model is able to identify solutions to the multiple-entry case that lead to far superior strategies to standard greedy approaches and millions of other strategies obtained through a structured random-sampling procedure.

The rest of this paper is organized as follows. Section 2 describes how the NFL regular season schedule is organized and the standard rules of NFL survivor pools. The model used to estimate NFL probabilities is then discussed in Section 3. The underlying optimization problem is formulated in Section 4. Optimization models for the single-entry and multientry variants are described in Sections 5 and 6, respectively, and an experimental evaluation is provided in Section 7. Finally, future work and a conclusion is provided in Section 8.

2. The NFL Season and Survivor Pool Details

The NFL season consists of 32 teams playing 16 games each. The schedule revolves around the concept of "weeks," where a given week of games are played between Thursday and Monday. Each team plays just one game per week and receives one “bye week” in which it does not play a game. The organization of the season by weeks has resulted in the development of survivor pools, where betting does not involve point spreads and stretches over multiple weeks of the season.

A survival pool is an iterative betting pool. It begins with each participant picking a team in week 1. Should the team chosen win its week 1 game, the participant continues to week 2 and is otherwise eliminated. The process continues until the end of the season; having correctly selected a winning team in each of the first $w$ weeks, a participant again must choose a team to win in week $w + 1$ and is eliminated if that team loses or ties. The only restriction is that a participant may not choose a team he chose earlier in the season. The winner of a survival pool is the participant who lasts the greatest number of weeks. If multiple participants last the full season, the winnings are split among them. It is
possible to win a survival pool, in the most extreme example, by correctly picking a week 1 team and seeing all other participants incorrectly choose in the first week. Generally speaking, the greater the number of participants, the longer a participant will have to last in order to win the pool. A participant’s choice for a team in any given week is made after all of the previous week’s games conclude and before any games in the next week begin.

The need to last many weeks into the season in order to win the survival pool, combined with the requirement that any given team be chosen no more than once, creates an obvious trade-off for the participant in making his decision each week; if he chooses a team to win in the current week, and it does, he advances to the next week of the pool at the expense of having lost that team as a choice in later weeks. Participants then have incentive to save good teams until later in the season instead of choosing them early; then again, an early season loss that eliminates the participant from the pool when he failed to use the best teams in his selections is an unhelpful result.

Assuming the participant in the survivor pool can reliably estimate the probabilities for any game of the season in any given week, this paper investigates strategies geared toward maximizing the probability of lasting the entire season. An additional complication in choosing whether or not to select a given team in a given week is that estimated win probabilities frequently change, sometimes substantially, over the course of a season.

Each participant is required to pay an entrance fee in order to participate in the pool. The money collected is then available to be won, with the payouts typically occurring according to some type of scale decided by the organizer. The payout scale can be dramatically different from survivor pool to survivor pool, but in all cases, lasting longer is certainly better and hence more lucrative. Unlike the study of Clair and Letscher (2007), which focuses on winning an NCAA tournament pool by selectively choosing underdogs that other participants will avoid, this study focuses on the probability of surviving the longest in an NFL survivor pool.

A complicating factor is that each participant may be allowed to pay an entrance fee for more than one entry. Each entry plays separately but all are commonly controlled by the participant paying the entrance fees. Typically, the number of entries per participant is capped, often to five or so. Expanding the analysis to multiple entries was mentioned as a possible generalization in the conclusion of the paper by Clair and Letscher (2007) and is considered in the present paper.

### 3. Probability Model

The academic literature on gambling on NFL outcomes focuses on the results of bets using previous game outcomes and point spreads. Sauer (1998) provides an extensive background on the characteristics and analysis of these betting markets. The focus of this paper is not on finding a precise mechanism for estimating win probabilities; rather, this paper focuses on modeling and identifying strategies for winning NFL survivor pools based on a given set of reliable probability estimates. A participant can derive his own win probability estimates via any approach and then adapt the computational models presented in the paper to identify winning strategies.

Clair and Letscher (2007) adapt team win probabilities based on Massey Ratings (Massey 2004). This paper assumes a similar technique for estimating the win probabilities, based on ESPN NFL Power Rankings, a weekly updated ranking of the 32 teams published by popular sports-broadcast network ESPN. Based on the rankings, a logistic regression model is employed to find, given the team that is playing at home and the relative ranks of the teams, the probability that the home team will win. The most important reason for choosing this mechanism for approximating win probabilities is that the computational models formulated in this paper require probability estimates for each remaining game in the season, which must be updated as the season progresses. Basing the win probabilities off another measure, such as point spreads, requires the current spread of each remaining NFL game in a season, updated weekly. To the best of our knowledge, such data are not available, at least not dating back to the 2002 NFL season. We also choose these data to estimate win probabilities because the expert rankings would account for all off-season roster changes (such as retirements or trades) in week 1 and then change weekly to reflect team performance, injuries, suspensions, player acquisitions, and coaching changes as the season progressed.

There are a variety of reasons why a participant’s estimate of win probabilities may change, most notably because of the performance of a team throughout the season. Consider the game in the last week of the 2013 season between the Houston Texans and Tennessee Titans. The Texans finished the 2012 season with a record of 12 wins and 4 losses (12–4), the third-best record in the entire NFL, while the Titans went 6–10. Expectations in week 1 of the 2013 season were that the Texans would win that game, but as the season progressed, with the Texans winning 2 of 13 games before playing the Titans, probability estimates for the Texans winning that game decreased substantially.

Historical ESPN NFL Power Rankings are available online (ESPN.com 2015) and can be downloaded for reproducibility. It is important to evaluate the accuracy of the probability model adapted in order to assess the validity of the survivor-pool strategies. The ESPN NFL Power Rankings became available in 2002, which was also the year that the current set of NFL teams and
A participant’s goal is to maximize the probability that one of his entries will outlast all other entries. Depending on \( n_e \), the number of consecutive weeks that an entry has to win in order to outlast all other entries varies. The underlying optimization problem faced by the participant is

\[
\max \Pr \left( \bigvee_{e \in E} S^e[t_1^e, t_2^e, \ldots, t_T^e] \right) \\
\text{subject to } t_e^{w_1} \neq t_e^{w_2}, \quad \forall w_1, w_2 \in \{1, \tilde{w}\}, w_1 \neq w_2.
\]

The optimization problem (1) seeks to maximize the probability that at least one of the participant’s entries will survive until the target week \( \tilde{w} \), subject to a set of constraints enforcing the condition that each entry cannot choose the same team in different weeks, per the rules of survivor pools.

There are a variety of complicating factors associated with identifying the optimal solution of the underlying optimization problem (1), the most prominent of which are the following.

1. **Factor F1**: The actual real-world probabilities are incomputable and change throughout the season, whereas the participant must choose one team per week.

2. **Factor F2**: The events are highly dependent on one another.

Factor F1 makes the optimal solution incomputable. Therefore, one must rely on estimates of the probabilities that, as discussed earlier, can be calculated in a variety of ways. Additionally, the participant must decide on a team at the start of each week. This places an onus on him to plan in advance, but, because the probabilities will change throughout the year, it may be beneficial not to plan the entire season in a given week. This will be incorporated in the computational models.

Factor F2 arises because mutual dependence of the events exists between the teams chosen by different entries in a given week and from outcomes in one week to another. For the single-entry case, the dependence only appears from week to week, and the probability estimates that are updated from week to week allow for modeling the effect of one week’s performance on the next week’s performance, as reflected in the relative ranking of the teams. For the multiplex-entry case, the calculation of the probabilities is more involved and requires highly nonlinear objective function terms. These appear because the model must take into account that a participant can select team \( t \) in week \( w \) for one entry and team \( a_\tilde{w}(t) \) in another entry in the same week (and this may be optimal).

5. **The Single-Entry Model**

The single-entry case is discussed first, where an optimization model is formulated that seeks to maximize
the probability of winning until a target week \( \tilde{w} \). For \( t \in T \) and \( w \in [1, \tilde{w}] \), define binary variable \( x_{w,t} \) that indicates whether team \( t \) is selected in week \( w \) by the single entry. Optimization problem (1) can then be modeled as a (nonlinear) binary IP:

\[
\max_{x_{w,t} \in \{0,1\}} \prod_{w=1}^{\tilde{w}} \left( \sum_{t \in T} p_{w,w,t} x_{w,t} \right)
\]

subject to

\[
\sum_{t \in T} x_{w,t} = 1, \quad \forall w \in [1, \tilde{w}],
\]

\[
\sum_{w=1}^{\tilde{w}} x_{w,t} \leq 1, \quad \forall t \in T.
\]

(2)

The objective function seeks to maximize the probability of winning in each week until the target week, subject to constraints enforcing that each team is selected at most once. For a fixed \( w \in \tilde{W} \), the first set of constraints require that exactly one of the variables \( x_{w,t} \) equals 1, resulting in \( \sum_{t \in T} p_{w,w,t} x_{w,t} \) evaluating to the probability that the team chosen by the participant wins in that week. Multiplying these terms results in the probability that the entry survives until \( \tilde{w} \). Optimization problem (2) assumes full knowledge of \( p_{w,w,t} \); however, these probability estimates are only known to the participant in the week \( w \). For example, in the first decision period, the participant only knows \( p_{1,w,1} \) and must rely on these estimates when making the decision in week 1, which is based on incomplete information about how the teams are projected to perform in future games.

This can be handled computationally by breaking the participant’s decisions into stages. The natural stages to consider are the \( \tilde{w} \) weeks, where in each stage an optimization problem will be solved and the selection of one team to pick for that week is made. To account for the shifting probabilities throughout the season, a parameter \( L \in [1, \tilde{w}] \) will represent the number of weeks to look ahead for planning. Namely, \( L = 1 \) corresponds to considering only the current week when making the decision in each stage, while, at the other end of the spectrum, if \( L = \tilde{w} - w \), the entire remaining season-long horizon is taken into account when making the decision in stage \( w \).

The model is formulated as follows. Suppose the participant is making a decision in week \( w \). The decisions in all stages \( w < w' \) to select teams \( t_{w'} \) must be used to restrict the decision in week \( w' \). For a given \( L \), the objective in stage \( w \) will be to maximize the probability of the entry surviving in the pool until week \( \tilde{L} = \min\{w' + L - 1, \tilde{w}\} \) given the decisions made in previous weeks. In the optimization problem, and thereby in the optimal solution, a set of teams to select in weeks \( w', \ldots, \tilde{L} \) will be determined, yet only the decision corresponding to the current week \( w' \) will actually be made, with all remaining decisions left for future weeks.

The optimization problem (3) in week \( w' \) with an \( L \)-week look ahead can then be formulated as follows, where \( t_{w'} \) is passed as input from previous stages to subsequent stages, where \( x_{w',w,t} \) indicates whether team \( t \) is chosen in week \( w' \) for week \( w' \):

\[
\max_{x_{w',w,t} \in \{0,1\}} \prod_{w=1}^{L} \left( \sum_{t \in T} p_{w',w,t} x_{w',w,t} \right)
\]

subject to

\[
\sum_{t \in T} x_{w',w,t} = 1, \quad \forall w \in [w', \tilde{L}],
\]

\[
\sum_{w=w'}^{L} x_{w',w,t} \leq 1, \quad \forall t \in T,
\]

\[
x_{w',w,t} = 0,
\]

\[
\forall w \in [w', \tilde{L}], \quad t \in \{t_{w'}: w \in [1, w' - 1]\},
\]

\[
t_{w'} = \sum_{t \in T} t \cdot x_{w',w,t}.
\]

(3)

The objective function maximizes the probability of selecting a winning team in each of the weeks \( w \in [w', \tilde{L}] \). The constraints, in order, enforce that (1) exactly one team is selected in each week in the planning horizon, (2) each team is selected at most once over the course of the weeks in the planning horizon, (3) no team previously selected can be selected in the planning horizon, and (4) the team selected in the current week \( w' \) (\( t_{w'} \)) is equivalent to the team whose corresponding binary variable in the week takes value 1.

This model allows for changing decisions when more information is made available. In particular, in week \( w' \), a plan for the next \( \tilde{L} \) weeks is made, but only one team is actually selected—the team for which \( x_{w',w',t} = 1 \). For example, suppose \( L > 1 \) and \( w' < n_{w'} \), the team \( t' \) for which \( x_{w',w'+1,t'} = 1 \) of the stage \( w' \) optimization problem will not necessarily be the team chosen in week \( w' + 1 \). The team will certainly be available and can be chosen, but should probabilities change and a better decision become available in subsequent weeks, the decision is free to be altered. The model simply lays out a plan for selections in the current and future weeks. The current week selection must be made, but if the participant wins that week and survives to the subsequent week, a different choice than was laid out in previous plans can still be made. Indeed, if the win probability estimates change substantially, the plan will be modified by future optimization models.

To implement a practical computational solution to solve optimization problem (3), the problem is linearized in order to arrive at a binary linear programming model by applying the log function to the objective function. Because the objective function is the product of terms, each of which will be greater than 0 because some team is chosen for each week in the planning horizon, and the objective is to maximize this function, the concave log function can be applied to the
objective function, resulting in the function (OF). The optimal value will change, but the optimal solutions will remain optimal after this transformation because \( \log \) is an increasing, concave function:

\[
\sum_{w \in w'} \log \left( \sum_{t \in T} p_{w', w, t} x_{w', w, t} \right) = \sum_{w \in w'} \log \left( \sum_{t \in T} (p_{w', w, t}) x_{w', w, t} \right).
\]

(OF)

Furthermore, because each of the internal sums will amount to exactly one of the probability being chosen, the logarithm can be distributed within the sum, resulting in the equivalent objective function assuming that only those variables corresponding to teams that do not have byes are defined (or some very small probability is chosen for teams in weeks in which they do not play). The model proposed for the single entry is thereby given by the parametrized \( \bar{w} \)-stage binary linear integer programs (3) with the modified objective function (OF). Note that the problem reduces to the maximum weight bipartite matching problem with the two shores of the graph corresponding to the teams and weeks, respectively, and the weight of an edge to the logarithm of the win probability.

6. The Multiple-Entry Model

Many NFL survivor pools allow participants to control multiple entries, typically on the order of 5 or so. This requires a more complex computational model, which, unfortunately, does not admit a simple linearization. The model must take into account that different entries could choose opponents in a given game. For example, suppose \( \bar{w} = 2 \) and that a participant has four entries. An optimal strategy is to choose different teams in week 1, one team per entry, with pairs of teams playing one another. Exactly two entries will then proceed to week 2, and again, choosing opponents will ensure that one entry survives.

A further complication is that the model requires the calculation of the union of the set of mutually dependent events and so relies on the principal of inclusion-exclusion (PIE), which can be stated as follows. Let \( A_1, \ldots, A_n \) be arbitrary events in a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Then,

\[
\Pr \left( \bigcup_{i=1}^n A_i \right) = \sum_{k=1}^n (-1)^{k-1} \sum_{(i_1, \ldots, i_k) \in \binom{[n]}{k}} \Pr (A_{i_1} \cap \cdots \cap A_{i_k}),
\]

where \( A_i = \bigcap_{t \in \mathcal{J}} A_i \) and \( \binom{[n]}{k} \) is the family of \( k \) element subsets of \( \{1, \ldots, n\} \). This allows a union of the probability of events to be recast as a sum of the intersection of all subsets of those events.

The objective function with multiple entries will still assume an \( L \)-stage look ahead in which decisions in the current week will be transferred as input to the model in subsequent weeks. The set of events under consideration in a given week \( w' \) will be \( A_i := S'[t'_w, t'_{w+1}, \ldots, t'_{w+L-1}] \).

The PIE decomposes the probability calculation into a sum of joint probabilities for each possible subset of \( E \), the set of entries. Let \( E' \) be any subset of \( E \) of size \( b \), where, for the moment and without loss of generality, assume that \( E' = [1, b] \). Then \( \Pr(A_{E'}) \) corresponds to the probability that each team in the set \( \bigcup_{i=1}^b \bigcup_{w \in [w', w'+L-1]} (t'_i) \) wins. Formulating this in the context of an optimization problem requires ensuring that the probability evaluates to 0 if two entries select opponents in any week within the planning horizon, as well as ensuring that each team is only calculated in the probability once—if two entries select the same team, then the probability calculation should only consider that team once.

The expanded variable set for the case of multiple entries is

- \( x_{w', w, t} \in \{0, 1\} \): indicates whether entry \( e \) plans to select team \( t \) in week \( w \) when planning in week \( w' \);
- \( z_{w'} \in T \): indicates the team that entry \( e \) selects in week \( w \);
- \( \alpha_{w', w} \in \{0, 1\} \): indicates whether the teams chosen by entry \( e_1 \) and \( e_2 \) in week \( w \) are opponents;
- \( c_{w', w} \in \{0, 1\} \): indicates whether the teams chosen by entry \( e_1 \) and \( e_2 \) in week \( w \) coincide.

Then, with the above integrality constraints, the optimization problem can be cast as

\[
\text{max} \sum_{k=1}^{n_e} (-1)^{k-1} \sum_{I \in \binom{[n_e]}{k}} \left( \prod_{e \in I} \prod_{t \in T} \left( 1 - \alpha_{w', w} \right) \right)
\]

\[
\left( \prod_{e \in I} \prod_{t \in T} \left( 1 - \prod_{e' \in I} \left( 1 - c_{w', w} \right) \right) \right)
\]

\[
\left( 1 - \sum_{t \in T} p_{w', w, t} x_{w', w, t} \right) + \sum_{t \in T} p_{w', w, t} x_{w', w, t} \}
\]

s.t.

\[
\sum_{t \in T} x_{w', w, t} = 1, \quad \forall e \in E, w \in [w', \bar{L}],
\]

\[
\sum_{w \in w'} x_{w', w, t} \leq 1, \quad \forall e \in E, t \in T,
\]

\[
(z_{w'} = t) \leftrightarrow (x_{w', w, t} = 1), \quad \forall e \in E, t \in T, w \in [w', \bar{L}],
\]

\[
x_{w', w, t} = 0, \quad \forall e \in E, w \in [w', \bar{L}],
\]

\[
t'_w = \sum_{t \in T} t \cdot x_{w', w, t}, \quad \forall e \in E, t \in T,
\]

\[
x_{w', w, t} + x_{w', w, t} = 1, \quad \forall e_1, e_2 \in E, t \in T, w \in [w', \bar{L}],
\]

\[
o_{w', w} \leq \sum_{t \in T} \left( x_{w', w, t} + x_{w', w, t} = 2 \right),
\]

\[
(z_{w'} = z_{w'}) \leftrightarrow c_{w', w} \leq 1,
\]

\[
\forall e_1, e_2 \in E, w \in [w', \bar{L}],
\]

\[
(e_1 < e_2, t \in T, w \in [w', \bar{L}]),
\]

(4)
**Theorem 1.** Constraint programming (CP) model (4) accurately models a participant’s survivor pool optimization problem, seeking to find the selection of teams for \( n_e \) entries that maximizes the probability of at least one entry surviving through week \( L \), starting in week \( w' \).

**Proof.** Consider the set of constraints sequentially. The first two constraints ensure that each entry selects some team in each week and that an entry cannot pick a given team in more than one week. The next constraints enforce that if \( z'_{w} = 1 \) if and only if \( x'_{w',w,t} = 1 \), and the following constraints enforce that the teams already selected by an entry before week \( w' \) are not selected again within the current planning horizon. The following constraints select the teams chosen to be passed as input to subsequent models. The next two constraints link the \( o \) variables with the \( x \) variables. The first set of constraints enforces that if two entries \( e_1 \) and \( e_2 \) pick the same team in a given week \( w \), then the corresponding \( o_{w_1}^{e_1,e_2} \) variable is equal to 1. The following constraints enforce that if \( o_{w_1}^{e_1,e_2} = 1 \), then, for at least one pair of opponents, they must be chosen by the entries \( e_1 \) and \( e_2 \) in week \( w \). The right-hand side of this constraint counts the number of times that the equality within the parentheses is satisfied. Finally, the last constraints link the \( z \) variables with the \( c \) variables.

Regarding the objective function, the outer sum \( \sum_{k=1}^{n_k} \) and the first inner sum \( \sum_{t \in T} \) are an instantiation of the PIE, so it suffices to show that for a fixed \( I \subseteq E \), the term within the summation properly calculates the joint probability of all of the entries successfully surviving up to week \( L \).

The term \( \prod_{w=w'}^{L} \prod_{e_1,e_2 \in T}(1-o_{w_1}^{e_1,e_2}) \) evaluates to 0 if and only if one of the terms \( o_{w_1}^{e_1,e_2} \) equals 1. This variable indicates that entries \( e_1 \) and \( e_2 \) have selected opponents in week \( w \), thereby making the probability vacuous that both entries will survive the entire span of weeks, as desired. If all entries select different teams in all of the weeks considered, this entire term becomes 1, making the term irrelevant in the final evaluation of the objective function.

The remaining portion of the objective function computes the product of the probabilities of the teams selected by any entry. This is accomplished via the two terms separated by the plus sign. For a fixed entry \( e \) and week \( w_1 \), \( (1-\prod_{e_1 \in I}(1-o_{w_1}^{e_1,e})) \) evaluates to 1 if some entry \( e' \in I', \) indexed lower than \( e \) (i.e., \( e' < e \)), selects the same team as \( e \) in week \( w \) and evaluates to 0 otherwise. In the former case, the second two lines in the objective function evaluate to

\[
1 \cdot \left( 1 - \sum_{t \in T} p_{w',w,t} x_{w',w,t}^{e} \right) + \sum_{t \in T} p_{w',w,t} x_{w',w,t}^{e} = 1.
\]

This is desired because the selected team has already been selected by some other entry so the probability is already incorporated in the product. In the latter case, this evaluates to

\[
0 \cdot \left( 1 - \sum_{t \in T} p_{w',w,t} x_{w',w,t}^{e} \right) + \sum_{t \in T} p_{w',w,t} x_{w',w,t}^{e} = \sum_{t \in T} p_{w',w,t} x_{w',w,t}^{e},
\]

which coincides with the probability that the team selected by entry \( e \) for week \( w \) will win. The product of these terms is therefore the probability of each team selected over the planning horizon by some entry in \( I \) winning, as desired. Q.E.D.

7. Experimental Results

All code is written in C++ (gcc version 4.8.2) and run in Ubuntu 14.04.1 LTS on a machine with an Intel® Core™ i7-4770 CPU @ 3.40 GHz processor and 32 GB of memory. The solvers used are IBM ILOG CPLEX 12.6 for IP models and IBM ILO CPO 12.6 for CP models.

7.1. Single-Entry Results

To provide an evaluation of the strategies identified, random strategies are generated for comparison. Each week, a team, among the set of teams still available, \( T' \), is selected based on a probability distribution scaled by the win probabilities. In particular, for week \( w \), let \( p_{\min} \) be the minimum win probability of any team in \( T' \), let \( q \) be calculated as \( \tilde{q}_i = \tilde{q}_i / \tilde{Q} \). These probabilities resulted in strategies that were far worse than those strategies generated by redefining \( T' = T' \cap \{ t \mid p_{w,t} \geq 0.5 \} \) (i.e., using only those available teams that are favored), and so the latter was used in the experiments.

Additionally, a one-week look-ahead strategy is identical to the participant picking the team, among those still available, with the highest chances of winning in each week. Since this is likely a standard strategy used by some participants in survivor pools, it is also included for comparison.

Finally, for the single-entry case, it is possible to identify the Ideal strategy. Consider model (3) with \( w' = 1, L \in \{1, \ldots, 17\} \) and, instead of \( p_{1,w,t} \) use probabilities \( p_{w,w,t} \), which are the win probabilities for the teams at the start of week \( w \). These probabilities cannot be used in identifying a strategy because \( p_{w,w,t} \) is not known to the participant until week \( w \); however, if these probabilities were known, the participant could make the optimal selection. No strategy can have a survival probability surpass the survival probability of Ideal, but the closer a strategy gets, the better.

First, a comparison of strategies for only the most recently completed NFL season (2014) is presented. Figure 1(a) compares the survival probability of the eight-week look-ahead strategy with the one-week
Figure 1. Comparison with Simulated Strategies for 2014 NFL Season

(a) One million simulated strategies

(b) One million simulated strategies

Figure 2. Comparison of Increasing Look-Ahead Strategies for 2014 NFL Season

(a) Survival probability

(b) (scaled) Survival probability

look-ahead strategy and one million randomly generated strategies. Each line corresponds to a strategy. The x axis corresponds to the number of weeks into the season, and the y axis shows the probability, in log scale, of surviving until that week given the employment of the given strategy. The plot shows the survival probabilities of the randomly generated strategy with the worst, max, and average survival probability. The single strategy that was determined by looking eight weeks in advance had a higher probability of survival for the full season than both the 1000000 Sim(Max) and greedy (or one-week look-ahead) strategies; it even approaches the success of Ideal. The one-week look-ahead strategy is competitive, though inferior to 1000000 Sim(Max).

Figure 1(b) shows another comparison with randomly generated strategies. The plot depicts the scaled survival probability, calculated each week as the ratio of the survival probability of the given strategy until that week divided by the probability of the Ideal strategy in that week. A line is included for the Ideal strategy, one-week look-ahead strategy, eight-week look-ahead strategy, and maximum survival probability strategy for an increasing number of randomly generated strategies—from one to one million, in powers of 10. The plot depicts how the survival probability through any given week increases as more entries join the survival pool. It also elucidates how the eight-week look-ahead strategy dominates the other strategies only when progressing into the later part of the season, which is necessary as the number of entries in the pool grows. If a 2014 pool had a relatively small number of entries, the greedy one-week look-ahead strategy may have performed better because, until week 10, the survival probability was higher for the one-week look-ahead strategy. However, in larger survival pools it is often necessary to survive the entire season, and so a longer-term planning horizon would have been more desirable.

The reason for choosing $L = 8$ is addressed computationally next. Figure 2(a) depicts the survival probabilities for $L \in \{1, 2, 5, 8, 11, 14, 17\}$, and Figure 2(b) depicts the same data but scaled based on Ideal. These
plots further elucidate the strengths and weakness of a greedy strategy. When seeking a strategy to survive seven or fewer weeks, the greedy strategy works well. As the season progresses, this strategy eventually is dominated by other strategies, ending with a final survival probability approximately half that of the eight-week look-ahead strategy. The eight-week look-ahead strategy became the best strategy to adopt if the participant anticipates needing to last more than halfway through the season.

These results conform in the aggregate, when evaluated over the most recent seasons. Figures 3(a) and 3(b) depict the scaled surviving probability per week, averaged over 2003–2008 and 2009–2014, respectively. In the more recent NFL seasons, it is apparent that a long-term strategy is superior to the greedy one-week look-ahead strategy. This can be attributed to the fact that the probability model has more data for training. For any given year, the probability model is calibrated using only the previous set of years, so as the years progress, more input data are accumulated. Also, ESPN started publishing their rankings in 2002, and the ESPN NFL Power Rankings have become more accurate in predicting future game outcomes. Considering the 100 highest probability predictions from 2003 to 2008, 83 are correctly predicted; from 2008 to 2014, 93 are correctly predicted. Because the optimization models for survivor pools seek to select the teams most likely to win, it is in this region that the probability estimates should be most accurate.

Figure 3 shows that the greedy one-week look-ahead strategy performs well, as it does for the 2014 season alone, when the participant is concerned with surviving only a few weeks. As the season progresses, however, the probability of surviving is only increased by looking further into the future. Looking the full 17 weeks into the future does not fare well though. Looking partway through the season yields the superior survival probability. In the more recent seasons, when the probability model is more accurate, an eight-week look ahead yields the best survival probability starting once the survivor pool winner reaches at least week 13.

7.2. Multiple-Entry Results
Unlike the single-entry case where each model is solved in a fraction of a second, the time required to solve CP model (4) is significant. For example, solving all stages for the multientry model with \( n_e = 2 \) and \( L = 2 \) takes over 20 minutes, and the run times increase exponentially as the parameters increase. Therefore, a time limit of 100 seconds per stage is imposed (1,700 seconds total for all stages), and the best solution found by that time is used for the analysis.

Additionally, since the search for solutions will be limited by time, it is of interest to investigate whether adding constraints to (4) can help in finding high-quality solutions quickly. Three constraints investigated by the authors were (1) preventing the entries from picking the same team in a given week, (2) restricting the set of entries from picking opponents in the same week, and (3) allowing entries only to select teams with win probabilities greater than 0.5. These restrictions can be added for only the current week or for the entire planning horizon. Through preliminary experimentation, it was determined that limiting the entries to only choose teams favored to win throughout the entire planning horizon and adding constraints that restrict any pair of entries from picking opponents in the current week led to the best results. That second restriction only impacts the set of available solutions when the probability of winning is exactly 50%, but, even when all win probabilities are not 50%, adding redundant constraints can affect search and solver decisions and, ultimately, the best solutions found. The best configuration added the following additional constraints on (4): (A) \( x_{w,w',t} = 0, \forall w \in [w',\bar{L}] \) for which \( p_{w,w',t} < 0.5 \), and (B) \( a_{w,t}^{c_1,c_2} = 0, \forall e, c_1, c_2 \in E, t \in T, w \in [w',\bar{L}] \). As an example, for the 2014 NFL season with
Table 2. 2014 NFL Season Survival Probabilities for Five Entries

<table>
<thead>
<tr>
<th>L</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
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<th>15</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.999</td>
<td>0.985</td>
<td>0.950</td>
<td>0.874</td>
<td>0.780</td>
<td>0.656</td>
<td>0.546</td>
<td>0.429</td>
<td>0.346</td>
<td>0.277</td>
<td>0.196</td>
<td>0.148</td>
<td>0.106</td>
<td>0.076</td>
<td>0.058</td>
<td>0.040</td>
<td>0.030</td>
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<tr>
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<td>0.947</td>
<td>0.869</td>
<td>0.775</td>
<td>0.652</td>
<td>0.540</td>
<td>0.427</td>
<td>0.343</td>
<td>0.275</td>
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<td>0.147</td>
<td>0.104</td>
<td>0.076</td>
<td>0.057</td>
<td>0.040</td>
<td>0.030</td>
</tr>
<tr>
<td>5</td>
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<td>0.974</td>
<td>0.943</td>
<td>0.877</td>
<td>0.802</td>
<td>0.692</td>
<td>0.591</td>
<td>0.493</td>
<td>0.417</td>
<td>0.336</td>
<td>0.241</td>
<td>0.191</td>
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<td>0.080</td>
<td>0.054</td>
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<tr>
<td>8</td>
<td>0.992</td>
<td>0.967</td>
<td>0.934</td>
<td>0.857</td>
<td>0.788</td>
<td>0.643</td>
<td>0.565</td>
<td>0.482</td>
<td>0.411</td>
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<td>0.247</td>
<td>0.200</td>
<td>0.150</td>
<td>0.116</td>
<td>0.087</td>
<td>0.059</td>
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<tr>
<td>11</td>
<td>0.998</td>
<td>0.973</td>
<td>0.917</td>
<td>0.853</td>
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</tr>
<tr>
<td>14</td>
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<td>0.793</td>
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<tr>
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<td>0.604</td>
<td>0.500</td>
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<td>0.273</td>
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<td>0.122</td>
<td>0.095</td>
<td>0.072</td>
<td>0.049</td>
<td>0.038</td>
</tr>
</tbody>
</table>

$n_e = 5$ and $L = 5$, adding no additional restrictions yields a solution with survival probability 0.01902, but with these imposed restrictions, the solution obtained has a survival probability of 0.04304. This configuration was therefore fixed for the remainder of the experimental results.

First, an analysis of how many weeks to look ahead in the season is provided for the 2014 NFL season and by aggregating all seasons. Table 2 reports the survival probabilities in the 2014 NFL season, given $L$ week look-ahead strategies. If surviving only a few weeks is the goal, then a single-week look-ahead strategy works well, but, as in the case of one entry, if more weeks are necessary to win the survivor pool, a more long-term strategy should be adopted.

Figure 4 depicts a plot of the probabilities in Table 2, scaled by the greedy one-week look-ahead strategy (for the 2014 NFL season) and then the same data averaged over all seasons in the test set, scaling each year by the greedy one-week look-ahead strategy in that year. These plots more readily display the advantage that a participant can get by planning more than one week in the future, and they show that adopting approximately a half-season look ahead yields strategies with the highest survival probability. We therefore fix $L = 8$ for the remaining experiments.

Figure 5 depicts the amount by which, for the 2014 NFL season, the survival probability grows as a participant adds entries. Figure 5(a) shows the survival probability per week for $n_e = 1, 2, \ldots, 5$, and Figure 5(b) shows the same data but as a percent increase over the greedy one-week look-ahead strategy, indicating that the survival probability more than triples when the number of entries is increased from 1 to 5, increasing the season-long survival probability from 1.4% to 4.6%.

A comparison to one million random strategies is provided in Figure 6. Five million single-entry strategies were generated and broken into groups of five, resulting in one million strategies, to simulate one million competitors. Figure 6 depicts the minimum, average, and maximum survival probabilities over all competitors. The survival probabilities per week for the single-entry and five-entry participant strategies with $L = 8$ are included, along with the one-entry Ideal strategy. The plot indicates that the eight-week look-ahead five-entry strategy is far better than the best five-entry random strategy, and it also shows that a participant should pay for extra entries if possible, since Ideal (with one entrant) has an even lower survival probability.

A brief discussion of the expected monetary gain is in order. Assuming that from year to year the outcomes of games are independent, with a yearly success probability of 0.04588 with five entries, a bettor reaches a probability of 0.506 of lasting the entire season at least once after 15 seasons. After 30 seasons, this probability increases to over 0.75. Since the money won in a survival pool far exceeds the entry fees, winning once will result in substantial monetary gain. For comparison, even using the best among the million random
strategies for five entries would take 32 years to reach a 0.5 probability of surviving the entire season at least once.

8. Conclusion

This paper investigates computational approaches for identifying strategies for NFL survivor pools. The authors formulate various optimization models and provide computational results comparing the strategies obtained. The experimental analysis determined that planning picks eight weeks into the future yields the highest probabilities for long-term survival—and hence victory—in the pool. Planning only partway through a season balances future uncertainty at the time that the decision maker must act. An estimate on the probability of a team winning a game 17 weeks into the future can change dramatically as the season progresses. Nonetheless, taking a myopic approach of only looking at the games in the current week does not allow for any future planning. In the context of NFL survivor pools, an eight-week look-ahead provides the right balance. This happens to be halfway through the NFL season; it is left for future work to investigate whether in other sports leagues—and more broadly, for other applications where probability estimates change over time—planning halfway through the horizon is the appropriate time window.

This paper lays the groundwork for identifying strategies for NFL survivor pools. Several interesting follow-up research questions arise, including the question of computational complexity. Consider the following decision problem: For a fixed \( n_e \) and \( L \), does there exist a selection of teams for which the probability of some entry surviving is greater than or equal to \( K \)? For \( n_e = 1 \), the problem reduces to a maximum weight bipartite matching problem, but the same reduction cannot be directly applied for general \( n_e \). Whether this problem is NP-hard or not is left for future work. Additionally, since the solution times are beyond reasonable limits, investigating dedicated solution approaches to the problem may lead to substantial improvements. Also, using different objective functions (e.g., maximizing the expected number of weeks) or considering variants of the problem for other leagues over longer time horizons to see whether looking halfway through the season is still best is of interest, though survival pool competitions as such seem to only exist for the NFL. Finally, using the optimization models developed in this paper to other forms of sports betting pools, and to other optimization problems with objectives on probability spaces with dependent events, may lead to novel optimization frameworks in a variety of contexts.

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References


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